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Chapter - 2

VECTOR CALCULUS

Scalars :→ A Physical quantity which has only magnitude and no direction.
Ex :→ Length, mass, temperature, time, speed density

Vectors :→ A physical quantity which both magnitude and fixed direction.
Ex :→ Displacement, velocity, acceleration force, momentum etc.

vectors denoted by a single letter with an arrow on it like $\vec{a}, \vec{b}, \vec{c}$ etc.

The magnitude (or length) of the vector $\vec{a}$ is denoted by $|\vec{a}|$

Unit vector :→
The vectors having unit magnitude (or length) is called unit vector.
or
Vectors whose magnitude is unity it is denoted by $\hat{a}$.

i.e $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ $\quad |\vec{a}| = \sqrt{\;}$

Dot or Scalar Product :→

The dot or scalar Product of two vectors $\vec{a}$ & $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ & is defined by

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta \Rightarrow ab\cos\theta \quad 0 \le \theta \le \pi$

where θ is the angle b/n the vectors $\vec{a}$ & $\vec{b}$

Properties

i] $i\cdot i = j\cdot j = k\cdot k = 1$

ii] $i\cdot j = j\cdot k = k\cdot i = 0$

iii) $\vec{a} = a_1 i + a_2 j + a_3 k$ $\quad \vec{b} = b_1 i + b_2 j + b_3 k$ then

1) $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

2) $\bar{a} \cdot \bar{a} = a_1^2 + a_2^2 a_3^2$

iv) $\bar{a} \cdot \bar{a} = |\bar{a}|^2$ or $\vec{a}^2 = |\bar{a}|^2 = a^2$

② Cross product or vector Product.

The cross product or vector product of two products $\vec{a}$ & $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ & is denoted by.

$$\vec{a} \times \vec{b} = |\bar{a}||\bar{b}| \sin\theta \cdot \hat{n}$$

$$\Rightarrow ab \sin\theta \, \hat{n}$$

Where $\hat{n}$ is the unit vector along the direction perpendicular to both $\vec{a}$ & $\vec{b}$

Properties. i] If $\vec{a} = 0$ or $\vec{b} = 0$ then $\bar{a} \times \bar{b} = 0$

2) $\bar{a} \times \bar{b} \neq \bar{b} \times \bar{a}$

$\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

3) $i \times i = j \times j = k \times k = 0$

$i \times j = k$, $j \times k = i$; $k \times i = j$

4) If $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$,
$\vec{c} = c_1 i + c_2 j + c_3 k$.

$$\bar{a}\,\bar{b}\,\bar{c} = \bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector Differential Operator :→

The vector differential operator is denoted by ∇ (read as del or Nabla) and is defined by

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

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$$\nabla \Rightarrow \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Where, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are called components of ∇ along i, j, k resp.

Gradient of a Scalar Point function :→

If $\phi(x, y, z)$ is a continuously differentiable Scalar function then gradient of ϕ is denoted to be $\nabla\phi$

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

Note :→

1) $\nabla\phi$ is a vector whose three components are $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$

2) ϕ is a Scalar point function while $\nabla\phi$ is a vector point function.

Divergence of a vector :→

Let F be any given continuously differentiable vector point function then divergence of $\vec{F}$ it is denoted by $\text{div}\,\vec{F}$ or $\nabla \cdot F$ (del dot F) is defined as

$$\text{div}\,\bar{F} = \nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) F$$

$$= i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

Note : $\nabla \cdot F$ is a Scalar quantity

Curl of a vector :→

Let F be any continuously differentiable vector point function then curl or rotation of F denoted by Curl F or $\nabla \times F$ (del cross F) is defined by

$$\text{curl } F = \nabla \times F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times F$$

$$= i \times \frac{\partial F}{\partial x} + j \times \frac{\partial F}{\partial y} + k \times \frac{\partial F}{\partial z}$$

Note : $\nabla \times F$ is a vector quantity.

Laplacian :→ If $\phi(x, y, z)$ is a continuously differentiable scalar function we define the Laplacian of ϕ as follows.

$$\text{Laplacian of } \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$ (scalar quantity)

Angle between two Surfaces :→

The angle between the two surfaces is defined to be equal to the angle between normals

If $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$ Where θ is the angle between the normals.

Directional Derivatives :→

If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is a given direction then, $D \cdot D = \nabla\phi \cdot \hat{n}$

$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$ is called Directional derivative of ϕ along $\hat{n}$.

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Find the unit vector normal to the surface at the indicated Points.

1) $x^2y + 2xz = 4$ at $(2,-2,3)$ HW ii) $xy^3z^2 = 4$ at $(-1,-1,2)$

Sol^n: Let $\phi = x^2y + 2xz$

The unit vector $\hat{n}$ normal to the surface $\phi = c$ is given by,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Ans: $\frac{-(i+3j+k)}{\sqrt{11}}$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(2xy+2z) + j(x^2+0) + k(2x)$$

At $(2,-2,3)$ $\Rightarrow i(2(2)(-2)+2(3)) + j(4) + k(2(2))$

$\Rightarrow -2i + 4j + 4k$

$\nabla\phi = 2[-i+2j+2k]$

$|\nabla\phi| = \sqrt{2^2(1+4+4)}$

$= \sqrt{4(9)}$

$= 2(3) \Rightarrow 6$

The required unit normal $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \Rightarrow \frac{2[-i+2j+2k]}{6} \Rightarrow \frac{-i+2j+2k}{3}$

2) Find the Directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $2i-j-2k$.

Sol^n: The unit vector in the direction $2i-j-2k$ is

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} \Rightarrow \frac{2i-j-2k}{\sqrt{4+1+4}} \Rightarrow \frac{2i-j-2k}{3}$$

The Directional Derivative is, $\boxed{DD = \nabla\phi\,\hat{n}}$ --- (1)

Given, $\phi = x^2yz + 4xz^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$\Rightarrow i(2xyz + 4z^2) + j(x^2z+0) + k(x^2y + 8xz)$

At $(1,-2,-1)$ $\nabla\phi \Rightarrow i(2(2)+4) + j(-1) + k(-2+8(-1))$

$\Rightarrow 8i - j - 10k$

(1) $\Rightarrow D.D = [8i-j-10k]\cdot(2i-j-2k)$

$$\Rightarrow \frac{16 + (1) + 20}{3} \Rightarrow \frac{37}{3}$$

2. Find D.D of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i - 3j + 6k$

Sol^n: The Directional derivative $[D.D = \nabla\phi \cdot \hat{n}]$ ---

The unit vector $\hat{n}$ on the direction is $2i-3j+6k$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} \Rightarrow \frac{2i-3j+6k}{\sqrt{4+9+36}} \Rightarrow \frac{2i-3j+6k}{7}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Rightarrow i(4z^3 - 6xy^2z) - (6x^2yz)j + k(12xz^2 - 3x^2y^2)$$

At $(2,-1,2) \Rightarrow 8i + 48j + 84k$

$$D.D = (8i + 48j + 84k) \cdot \frac{2i-3j+6k}{7}$$

$$\Rightarrow \frac{16 - 144 + 504}{7} = \frac{376}{7}$$

3. Find D.D of $\phi = x\log z - y^2 + 4$ at $(-1,2,1)$ in the direction of the vector $2i - j - 2k$.

Sol^n: The $D.D = \nabla\phi \cdot \hat{n}$ --- ①.

The unit vector $\hat{n}$ on the direction is $2i-j-2k$

$$\hat{n} = \frac{\bar{d}}{|\bar{d}|} = \frac{2i-j-2k}{\sqrt{4+1+4}} \Rightarrow \frac{2i-j-2k}{3}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Rightarrow i(\log z) + j(-2y) + k(x/z)$$

At $(-1,2,1)$, $\nabla\phi = i(\log 1) - j(2(2)) + k(-1/1)$

$$\nabla\phi = 0 - 4j - k$$

$$DD = -4j - k\left[\frac{2i-j-2k}{3}\right]$$

$$\Rightarrow \frac{0+4+2}{3} = \frac{6}{3} = 2$$

$\Rightarrow \dfrac{16 + (1) + 20}{3} \Rightarrow \dfrac{37}{3}$ //

2. Find D.D of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i - 3j + 6k$

Sol^n:- The Directional derivative $\boxed{D.D = \nabla\phi \cdot \hat{n}}$ --- ①

The unit vector $\hat{n}$ on the direction is $2i-3j+6k$.

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} \Rightarrow \frac{2i - 3j + 6k}{\sqrt{4+9+36}} \Rightarrow \frac{2i-3j+6k}{7}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Rightarrow i(4z^3 - 6xy^2z) - (6x^2yz)j + k(12xz^2 - 3x^2y^2)$$

At $(2,-1,2) \Rightarrow 8i + 48j + 84k$.

$$D.D = (8i + 48j + 84k) \cdot \frac{2i - 3j + 6k}{7}$$

$$\Rightarrow \frac{16 - 144 + 504}{7} = \frac{376}{7}$$ //

3. Find D.D of $\phi = x\log z - y^2 + 4$ at $(-1,2,1)$ in the direction of the vector $2i - j - 2k$.

Sol^n:- The D.D $= \nabla\phi \cdot \hat{n}$ --- ①.

The unit vector $\hat{n}$ on the direction is $2i - j - 2k$

$$\hat{n} = \frac{\bar{d}}{|\bar{d}|} = \frac{2i - j - 2k}{\sqrt{4+1+4}} \Rightarrow \frac{2i - j - 2k}{3}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Rightarrow i(\log z) + j(-2y) + k(x/z)$$

At $(-1,2,1)$, $\nabla\phi = i(\log 1) - j(2(2)) + k(-1/1)$

$$\nabla\phi = 0 - 4j - k$$

$$DD = -4j - k\left[\frac{2i - j - 2k}{3}\right]$$

$$\Rightarrow \frac{0 + 4 + 2}{3} = \frac{6}{3} = 2$$ //

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5] Find $\nabla\phi$, if $\phi = x^3+y^3+z^3-3xyz$ at the Point $(1,-1,2)$

Sol^n:- $\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$

$\nabla\phi = i(3x^2-3yz)+j(3y^2-3xz)+k(3z^2-3xy)$

At $(1,-1,2)$ $\nabla\phi = i(3-3(-2))+j(3-3(2))+k(12-3(-1))$

$\Rightarrow 9i-3j+15k.$

HW Given $\phi = x^2y+y^2z+z^2x$ find $\nabla\phi$

1. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$

Sol^n:- Equation of the two surfaces given by

$x^2+y^2-z=3$; $x^2+y^2+z^2=9$

Let $\phi_1 = x^2+y^2+z^2$ & $\phi_2 = x^2+y^2-z$

$\nabla\phi_1 = \dfrac{\partial\phi_1}{\partial x}i + j\dfrac{\partial\phi_1}{\partial y} + k\dfrac{\partial\phi}{\partial z}$; $\nabla\phi_2 = i\dfrac{\partial\phi_2}{\partial x} + j\dfrac{\partial\phi_2}{\partial y} + k\dfrac{\partial\phi_3}{\partial z}$

$\nabla\phi_1 \Rightarrow 2xi+2yj+2zk$ $\quad \nabla\phi_2 = 2xi+2yj-k$

At $\nabla\phi_1[2,-1,2] = 2(2)i+2(-1)j+2(2)k$

$\Rightarrow 4i-2j+4k$

$= 2(2i-j+2k)$

At $\nabla\phi_2[2,-1,2] = 2(2)i+2(-1)j-k$

$= 4i-2j-k.$

If θ is the angle between two normals

$\cos\theta = \dfrac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|}$

$\cos\theta = \dfrac{2(8+2-2)}{\sqrt{(36)(21)}}$

$\cos\theta = \dfrac{16}{6\sqrt{21}} \Rightarrow \dfrac{8}{3\sqrt{21}}$

$\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$ //

(2). Find the angle between the surfaces $x^2+y^2-z^2=4$ & $z=x^2+y^2-13$ at $(2,1,2)$

Sol^n:- The given equation of the two surface given by

$x^2+y^2-z^2=4$ & $z=x^2+y^2-13$

Let $\phi_1 = x^2+y^2-z^2$ $\quad \phi_2 = x^2+y^2-z$

we have,

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x} i + \frac{\partial\phi_1}{\partial y} j + \frac{\partial\phi_1}{\partial z} k$$

$$\Rightarrow 2x i + 2y j - 2z k$$

At $(2,1,2)$ $\Rightarrow 2(2)i + 2(1)j - 2(2)k$

$$= 4i + 2j - 4k.$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x} i + \frac{\partial\phi_2}{\partial y} j + \frac{\partial\phi_2}{\partial z} k.$$

$$\Rightarrow 2x i + 2y j - k.$$

At $(2,1,2)$ $\Rightarrow 2(2)i + 2(1)j - k$

$$\Rightarrow 4i + 2j - k$$

If θ is angle between two normals

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4i+2j-4k)(4i+2j-k)}{\sqrt{16+4+16}\ \sqrt{16+4+1}}$$

$$\Rightarrow \frac{16+4+4}{6(\sqrt{21})} \Rightarrow \frac{24}{6\sqrt{21}} \Rightarrow \frac{4}{\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right) //$$

(3) Find the angle between the surfaces $xy^2z = 3x+z^2$ & $3x^2-y^2+2z=1$ At $(1,-2,1)$

Sol^n:- The given eq^n of two surfaces given by

Let $\phi_1 = 3x^2-y^2+2z$ $\quad \phi_2 = 3x+z^2-xy^2z$

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$= i(6x) + j(-2y) + k(2)$$

At $(1,-2,1)$ $\nabla\phi_1 = 6(1)i - 2(-2)j + 2k$

$$= 6i + 4j + 2k$$

$$\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z} \Rightarrow [3-(y^2z)]i + (-2xyz)j$$

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$\Rightarrow i(3 - y^2z) + j(-2xyz) + k(2z - xy^2)$

At $(1,-2,1) \Rightarrow (3-4)i + j(-2(-2)) + k(2-4)$

$\nabla\phi_2 \Rightarrow -i + 4j - 2k.$

If θ is angle between two normals

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(6i+4j+2k)(-i+4j-2k)}{\sqrt{36+16+4}\ \sqrt{1+16+4}}$$

$$\Rightarrow \frac{-6+16-4}{\sqrt{56}\ \sqrt{21}} \Rightarrow \frac{6}{\sqrt{4\times14}\ \sqrt{21}} \Rightarrow \frac{6}{2\sqrt{14}\sqrt{21}} =$$

$$\cos\theta = \frac{3}{\sqrt{14}\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{294}}\right) ; \Rightarrow \cos^{-1}\left(\frac{3}{\sqrt{49\times6}}\right) = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right) //$$

1) Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

Soln :- Let $\phi = x^3+y^3+z^3-3xyz$

$\vec{F} = \text{grad}\ \phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\therefore \vec{F} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$ ---- (1)

$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[(3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k\right]$

$\nabla F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)[3x^2-3yz]\,i + (3y^2-3xz)j + (3z^2-3xy)k$

$\Rightarrow \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

div$\vec{F} = \nabla\cdot F \Rightarrow 6x + 6y + 6z \Rightarrow 6(x+y+z)$ //

Also, curl $\vec{F} = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$

$\Rightarrow i\{-3x+3x\} - j\{-3y+3y\} + k\{-3z+3z\}$

$\Rightarrow \nabla\times\vec{F} = 0$ //

2) If $F = \nabla(xy^3z^2)$ find $\text{div}\vec{F}$ and $\text{curl}\,\vec{F}$ at the Point $(1,-1,1)$

Sol^n:→ Let $\phi = xy^3z^2$

$\vec{F} \Rightarrow \text{grad}\,\phi = \nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$

$\vec{F} \Rightarrow (y^3z^2)i + (3xy^2z^2)j + (2xy^3z)k$ ----

$\text{div}\,\vec{F} = \nabla\vec{F} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)\{y^3z^2i + 3xy^2z^2j + 2xy^3z$

$\nabla\vec{F} \Rightarrow \dfrac{\partial}{\partial x}(y^3z^2) + \dfrac{\partial}{\partial y}(3xy^2z^2) + \dfrac{\partial}{\partial z}(2xy^3z)$

$\nabla\vec{F} \Rightarrow \quad 0 + 6xyz^2 + 2xy^3$

At $(1,-1,1) = \nabla\cdot\vec{F}$ or $\text{div}\,\vec{F} \Rightarrow 6(1)(-1)(1) + 2(1)(-1)^3$

$\text{div}\,\vec{F} = \boxed{\nabla\vec{F} = -8}$

$\text{curl}\,\vec{F} = \nabla\times\vec{F} \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

$\Rightarrow i(6xy^2z - 6xy^2z) - j(2y^3z - 2y^3z) + k(3y^2z^2 - 3y^2z^2)$

$\text{curl}\,\vec{F} = 0$ //

HW If $\vec{F} = (x+y+1)i + j - (x+y)k$ Show that $\vec{F}\cdot\text{curl}\,\vec{F} = 0$

3) Find $\text{div}\vec{F}$ & $\text{curl}\,\vec{F}$ at $(1,2,3)$ If $\vec{F} = 3x^2i + 5xy^2j + 5xyz^3k$

Sol^n:→ Given, $\vec{F} = 3x^2i + 5xy^2j + 5xyz^3k$

$\text{div}\,\vec{F} = \nabla\vec{F} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)(3x^2i + 5xy^2j + 5xyz^3k)$

$\Rightarrow \dfrac{\partial}{\partial x}(3x^2) + \dfrac{\partial}{\partial y}(5xy^2) + \dfrac{\partial}{\partial z}(5xyz^3)$

$\Rightarrow 6x + 10xy + 15xyz^2$

At $(1,2,3) \Rightarrow \nabla\vec{F} \Rightarrow 6(1) + 10(1)(2) + 15(1)(2)(3)^2$

$\Rightarrow 6 + 20 + 30(9) \Rightarrow 26 + 270$

$\boxed{\nabla\vec{F} \Rightarrow 296}$ //

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$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 & 5xy^2 & 5xyz^3 \end{vmatrix}$$

$$\Rightarrow i\{5xz^3 - 0\} - j\{5yz^3 - 0\} + k\{5y^2 - 0\}$$

$$= 5xz^3 i - 5yz^3 j + 5y^2 k.$$

At (1, 2, 3) $\nabla\times\vec{F} = [5(1)(3)^3]i - 5(2)(3^3)j + 5(2^2)k$

$$\Rightarrow 135i - 270j + 20k$$

4) Evaluate curl (curl $\vec{F}$) & Div (curl $\vec{F}$) if

$$\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$$

Sol^n: Given $\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$\Rightarrow i\{0 - y^2\} - j\{z^2 - 0\} + k\{0 - x^2\}$$

$$\nabla\times\vec{F} \Rightarrow -y^2 i - z^2 j - x^2 k$$

$$\text{curl}(\text{curl}\,\vec{F}) \Rightarrow \nabla\times(\nabla\times\vec{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$\Rightarrow i[0 + 2z] - j[-2x + 0] + k[0 + 2y]$$

$$\nabla\times(\nabla\times\bar{F}) \Rightarrow 2zi + 2xj + 2yk \;//$$

$$\text{Div}(\text{curl}\,\vec{F}) \Rightarrow \nabla\cdot(\nabla\times\bar{F}) \Rightarrow [i\,\partial/\partial x + j\,\partial/\partial y + k\,\partial/\partial z][-y^2i - z^2j - x^2 k]$$

$$\Rightarrow -0 - 0 - 0$$

$$\nabla\cdot(\nabla\times\bar{F}) \Rightarrow 0 \;//$$

Solenoidal and Irrotational Vectors

Solenoidal vectors :→ A vector whose divergence vanishes identically is called Solenoidal vector.
i.e F is solenoidal iff div F = 0. or $\nabla \cdot \vec{F} = 0$

Irrotational vectors :→ A vector whose curl vanishes identically is called an irrotational vector.
i.e F is irrotational iff $\nabla \times \bar{F} = 0$ or curl $\bar{F} = 0$.

① Show that $\vec{F} = (y+z)i + (z+x)j + (x+y)k$ is irrotational. Also find scalar function ϕ such that $\vec{F} = \nabla\phi$

Sol^n:→ Given,

$$\vec{F} = (y+z)i + (z+x)j + (x+y)k$$

We have to show that curl F = 0.

$$\text{curl } \bar{F} \Rightarrow \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & z+x & x+y \end{vmatrix}$$

$$\Rightarrow i[1-1] - j[1-1] + k[1-1]$$

$$\nabla \times \bar{F} = 0$$

∴ The function $\bar{F}$ is irrotational.

consider $\nabla\phi = \vec{F}$

$$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (y+z)i + (z+x)j + (x+y)k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y+z \qquad \therefore \phi = \int(y+z)\,dx + f_1(y,z)$$

$$\phi = xy + xz + f_1(yz) \quad \text{--- ①}$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = z+x \qquad \therefore \phi = \int(z+x)\,dy + f_2(x,z)$$

$$\phi = yz + xy + f_2(x,z) \quad \text{--- ②}$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = x+y \qquad \therefore \phi = \int(x+y)\,dz + f_3(x\,y)$$

$$\phi = xz + yz + f_3(xy) \quad \text{--- ③}$$

let us choose, arbitrary fun^n $f_1(y/z)$ $f_2(x,z)$ $f_3(xy)$
To choose $f_1(y,z)$ we look into the ② ③ & select terms which do not contain x ∴ it can be terms with y or z or y & z. Illy we have to choose $f_2(xz)$ from ① & ③, & $f_3(x,y)$ from ① & ②.

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choose $f_1(y,z) = yz$

$f_2(x,z) = xz$

$f_3(xy) = xy$

Thus the required $\phi = xy + yz + zx$ //

2) Find the value of the constant 'a' such that the vector field $\vec{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$

Sol^n:- Given, $\bar{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$

We have to show that curl $F = 0$.

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$\Rightarrow i[0-0] - j[(1-a)z^2 + 3z^2] + k(2x(a-2) - ax]$

$\Rightarrow -j[z^2 - az^2 + 3z^2] + k[2ax - 4x - ax]$

$\Rightarrow -j[4z^2 - az^2] + k[ax - 4x) = 0$

$\Rightarrow -j[4-a]z^2 + (a-4)xk = 0$

$\Rightarrow (a-4)z^2 j + (a-4)xk = 0$

$\boxed{a = 4}$

Now Consider, $\nabla\phi = \bar{F}$ Put $a=4$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (4xy - z^3)i + (2)x^2 j - 3xz^2 k$

$\frac{\partial\phi}{\partial x} = 4xy - z^3 \quad \therefore \phi = \int(4xy - z^3)dx + f_1(y,z)$

$\phi = 4\frac{x^2}{2}y - xz^3 + f_1(yz)$

$\phi = 2x^2y - xz^3 + f_1(y,z)$ - - - - (1).

$\frac{\partial\phi}{\partial y} = 2x^2 \quad \therefore \phi = \int 2x^2 dy + f_2(x,z)$

$\phi = 2x^2y + f_2(x,z)$ - - - (2).

$\frac{\partial\phi}{\partial z} = -3xz^2 \quad \therefore \phi = -\int 3xz^2 + f_3(xy)$

$\phi = -3xz^3 + f_3(x,y)$

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choose $f_1(y,z) = yz$
$f_2(x,z) = xz$
$f_3(xy) = xy$

Thus the required $\phi = xy + yz + zx$ //

2) Find the value of the constant 'a' such that the vector field $\vec{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$

Soln:- Given, $\bar{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$

We have to show that curl $F = 0$.

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$\Rightarrow i[0-0] - j[(1-a)z^2 + 3z^2] + k(2x(a-2) - ax]$

$\Rightarrow -j[z^2 - az^2 + 3z^2] + k[2ax - 4x - ax]$

$\Rightarrow -j[4z^2 - az^2] + k[ax - 4x) = 0$

$\Rightarrow -j[4-a]z^2 + (a-4)xk = 0$

$\Rightarrow (a-4)z^2 j + (a-4)xk = 0$

$\boxed{a = 4}$

Now Consider, $\nabla\phi = \bar{F}$ Put $a=4$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (4xy - z^3)i + (2)x^2 j - 3xz^2 k$

$\frac{\partial\phi}{\partial x} = 4xy - z^3 \quad \therefore \phi = \int(4xy - z^3)dx + f_1(y,z)$

$\phi = 4\frac{x^2}{2}y - xz^3 + f_1(yz)$

$\phi = 2x^2y - xz^3 + f_1(y,z)$ ---- (1).

$\frac{\partial\phi}{\partial y} = 2x^2 \quad \therefore \phi = \int 2x^2 dy + f_2(x,z)$

$\phi = 2x^2y + f_2(x,z)$ --- (2).

$\frac{\partial\phi}{\partial z} = -3xz^2 \quad \therefore \phi = -\int 3xz^2 + f_3(xy)$

$\phi = -3x\frac{z^3}{3} + f_3(x,y)$

$$\phi = -xz^3 + f_3(x,y) \quad \cdots (3)$$

Let $f_1(y,z) = 0$, $f_2(x,z) = -xz^3$, $f_3(x,y) = 2x^2y$

Thus $\phi = 2x^2y - xz^3$ //

(3). If $\vec{F} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$ is irrotational, find a, b, c such that $\vec{F} = 0$ & then find ϕ such that $\vec{F} = \nabla\phi$.

Given,

Sol^n: $\bar{F} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$

We have to show that curl $F = 0$.

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix} = 0$$

$\Rightarrow i[c+1] - j[1-a] + k[b-1] = 0$

$\Rightarrow c = -1$, $a = 1$, $b = 1$

Now, consider $\nabla\phi = \bar{F}$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = (x+y+z)i + (x+2y-z)j + (x-y+2z)k$

$\Rightarrow \frac{\partial\phi}{\partial x} = x+y+z \qquad \therefore \phi = \int (x+y+z)dx + f_1(yz)$

$$\phi = \frac{x^2}{2} + xy + xz + f_1(yz) \quad \cdots (1)$$

$\frac{\partial\phi}{\partial y} = x+2y-z \qquad \therefore \phi = \int (x+2y-z)dy + f_2(xz)$

$$\phi = xy + y^2 - yz + f_2(xz) \quad \cdots (2)$$

$\frac{\partial\phi}{\partial z} = x+cy+2z \qquad \therefore \phi = \int (x-y+2z)dz + f_3(xy)$

$$\phi = xz - yz + z^2 + f_3(xy) \quad \cdots (3)$$

Let $f_1(yz) = y^2 - yz + z^2$; $f_2(x,z) = \frac{x^2}{2} + xz + z^2$, $f_3(xy) = \frac{x^2}{2} + xy + y^2$

$\therefore$ Thus the required

$$\phi = y^2 - yz + z^2 + \frac{x^2}{2} + xz + z^2 + \frac{x^2}{2} + xy + y^2$$

$$\therefore \phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2$$

//

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4) Find the constants a, b & c such that $\vec{F} = (axy - z^3)i + (bx^2 + z)j + (bxz^2 + cy)k$ is irrotational.

Sol^n:→ Given, $\vec{F} = (axy - z^3)i + (bx^2 + z)j + (bxz^2 + cy)k$

We have to show that curl F = 0.

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & bx^2 + z & bxz^2 + cy \end{vmatrix} = 0$$

$$i[c-1] - j[bz^2 + 3z^2] + k[2bx - ax] = 0$$

$$i(c-1) - j(b+3)z^2 + k(2b-a)x = 0$$

$c = 1$, $b = -3$, $2b - a = 0$

$2b = a$

$2(-3) = a$

$\boxed{a = -6}$, $\boxed{b = -3}$, $\boxed{c = 1}$

5) Show that $\vec{A} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal & irrotational.

Sol^n:→ Given $\vec{A} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$

For $\vec{A}$ to be solenoidal $\nabla \cdot \vec{A} = 0$

$$\Rightarrow \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[(y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k\right]$$

$$\Rightarrow (-2 + 2x - 2x + 2)$$

$\boxed{\nabla \cdot \vec{A} = 0}$. Hence $\vec{A}$ is solenoidal.

For $\vec{A}$ to be irrotational $\nabla \times \vec{A} = 0$

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$\Rightarrow i[3x - 3x] - j[3y - 2z + 2z - 3y] + k[3z + 2y - 2y - 3z]$$

$\boxed{\nabla \times \vec{A} = 0}$ Hence $\vec{A}$ is irrotational.

6) Show that

$$\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2+y^2}$$ is solenoidal And irrotational

Sol^n:→ Given, $\vec{F} = \frac{xi+yj}{x^2+y^2} \Rightarrow \frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j$

For $\bar{F}$ to be solenoidal $\nabla \cdot \bar{F} = 0$

$$\Rightarrow \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[\frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j\right]$$

$$\Rightarrow \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$\Rightarrow \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)1 - y(2y)}{(x^2+y^2)^2}$$

$$\nabla\cdot\vec{F} \Rightarrow \frac{y^2 - \cancel{x^2}}{(x^2+y^2)^2} + \frac{x^2 - \cancel{y^2}}{(x^2+y^2)^2} \Rightarrow 0 \qquad \therefore \nabla\vec{F} = 0$$

$\therefore \vec{F}$ is solenoidal.

For $\vec{F}$ to be irrotational. curl $\vec{F} = 0. = \nabla\times\bar{F} = 0.$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/x^2+y^2 & y/x^2+y^2 & 0 \end{vmatrix}$$

$$i[0-0] - j[0-0] + k\left[y\left[\frac{-1(2x)}{(x^2+y^2)^2}\right] - x\left(-\frac{1(2y)}{(x^2+y^2)^2}\right)\right]$$

$$\Rightarrow k\left[-\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right]$$

$\therefore$ curl $F = 0 \Rightarrow \nabla\times\bar{F} = 0.$

Hence $\vec{F}$ is irrotational

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1) If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ Prove the followings.
i) $\nabla \cdot \vec{r} = 3$ ii) $\nabla \times \vec{r} = 0$ iii) $\nabla r^n = n r^{n-2} \vec{r}$

HW. iv) $\nabla^2 (r^n) = n(n+1) r^{n-2}$

Soln: Given, $\vec{r} = xi + yj + zk$

$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$ and differentiating partially w.r.to x we get.

$2r \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$ Also $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

i) $\text{div } r = \nabla \cdot \vec{r} \Rightarrow \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] (xi + yj + zk)$

$\Rightarrow \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

$= 1 + 1 + 1$

$\nabla \cdot \vec{r} = 3$ //

ii) $\nabla \times \vec{r} = 0$ or curl r

$$\nabla \times \vec{r} \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$= i\left\{ \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right\} - j\left\{ \frac{\partial z}{\partial x} - \frac{\partial x}{\partial x} \right\} + k\left\{ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right\}$

$= i(0) + j(0) + k(0)$

$\nabla \times \vec{r} = 0$

iii) $\nabla r^n = n r^{n-2} \vec{r}$

$\nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}$

$\Rightarrow i\, n r^{n-1} \frac{\partial r}{\partial x} + j\, n r^{n-1} \frac{\partial r}{\partial y} + k\, n r^{n-1} \frac{\partial r}{\partial z}$

$\Rightarrow n r^{n-1} \left\{ i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r} \right\}$

$\Rightarrow \frac{n r^{n-1}}{r} (xi + yj + zk)$

$\nabla r^n \Rightarrow n r^{n-2} \cdot \vec{r}$

$$\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2+y^2} \text{ is}$$

Soln:→ Given, $\vec{F} = \frac{x\hat{i}+}{x^2}$

For $\bar{F}$ to be

$\Rightarrow \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right.$

$\Rightarrow \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right.\right.$

$\Rightarrow (x^2+y^2)$

continue...

$|\bar{r}| = r$ find grad(div $\vec{r}/r$)

Integration

The process of finding an integral of a vector function is called vector integration. It is divided into three types:
i) Line integral ii) Surface integral iii) Volume integral

Line integral :→

An integral evaluated to a vector point function along a line between the points is called Line integral.

For a vector point function $\bar{F}$, The Line integral between the points A & B is given by

$$\int_C \bar{F} \cdot d\bar{r} = \int_A^B \bar{F} \cdot d\bar{r} \quad \text{Where } \bar{r} = xi + yj + zk$$

(it is standard formula for Positional vector
& $d\bar{r} = dxi + dyj + dzk$.

It is also used to find the total workdone in moving a particle from one point to another on applying an external force.

①. If $\vec{F} = xyi + yzj + zxk$ evaluate $\int_C \bar{F} \cdot d\bar{r}$ Where C is the curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$

Soln:→ We have

$$\vec{F} = xyi + yzj + zxk \quad \text{and Let } \bar{r} = xi + yj + zk$$

$$\& \; d\bar{r} = dxi + dyj + dzk$$

Consider, $\bar{F} \cdot d\bar{r} =$

But given

$\therefore \vec{F} \cdot d\bar{r}$

② If

iv) $\nabla^2 r^n = \left(i\,\partial^2/\partial x + j\,\partial/\partial y + k\,\partial/\partial z\right)(r^n)$

$\Rightarrow \sum \frac{\partial^2}{\partial x^2}(r^n)$

$\Rightarrow \sum \partial/\partial x\ \partial/\partial x\,(r^n)$

continue...

HW. If $\vec{r} = xi + yj + zk$ & $|\vec{r}| = r$ find grad$(\text{div}\ \vec{r}/r)$

soln:-

Vector Integration

A mathematical Process of finding an integral of a vector point function is called vector integration. it is divided into three types.

i) Line integral   ii) Surface integral   iii) volume integral

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soln:- We have

$\vec{F} = xyi + yzj + zxk$ and Let $\bar{r} = xi + yj + zk$

iv) $\nabla^2 r^n = \left(i\,\partial^2/\partial x + j\,\partial/\partial y + k\,\partial/\partial z\right)(r^n)$

$\Rightarrow \sum \frac{\partial^2}{\partial x^2}(r^n)$

$\Rightarrow \sum \partial/\partial x \; \partial/\partial x \,(r^n)$ continue....

H.W. If $\vec{r} = xi + yj + zk$ & $|\vec{r}| = r$ find grad $(\operatorname{div} \vec{r}/r)$

soln:-

Vector Integration

A mathematical Process of finding an integral of a vector point function is called vector integration. it is divided into three types.

i) Line integral ii) Surface integral iii) Volume integral

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①. If $\vec{F} = xyi + yzj + zxk$ evaluate $\int_C \bar{F}\cdot d\bar{r}$ Where C is the curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$

soln:- We have

$\vec{F} = xyi + yzj + zxk$ and Let $\bar{r} = xi + yj + zk$

& $d\bar{r} = dxi + dyj + dzk$

Consider, $\bar{F}\cdot d\bar{r} = (xyi + yzj + zx^2k)(dxi + dyj + dzk)$

$= xy\,dx + yz\,dy + zx\,dz$

But given that $x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = t^3 \Rightarrow dz = 3t^2\,dt$

$\therefore \vec{F}\cdot d\bar{r} := t(t^2)\,dt + t^2 t^3\, 2t\,dt + t^3 t\, 3t^2\,dt$

$= t^3\,dt + 2t^6\,dt + 3t^6\,dt$

$= (t^3 + 5t^6)\,dt$

$\therefore \int \bar{F}\cdot d\bar{r} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt$

$\Rightarrow \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1} \Rightarrow \left[\frac{1}{4} + \frac{5}{7} - \frac{1}{4} + \frac{5}{7}\right]$

$\int \bar{F}\cdot d\bar{r} = 10/7$ //

(2). If $\vec{F} = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve given by $x = t$, $y = t^2$, $z = t^3$

Soln:→ Given, $\vec{F} = (3x^2+6y)i - 14yzj + 20xz^2k$

Let $\bar{r} = xi + yj + zk$

$d\bar{r} = dxi + dyj + dzk$.

$\bar{F}\cdot d\bar{r} = [(3x^2+6y)i - 14yzj + 20xz^2k]\, dxi + dyj + dzk$.

$\Rightarrow (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$

Given that $x = t$ $dx = dt$ $(0,0,0)$ to $(1,1,1)$

$y = t^2$ $dy = 2t\,dt$ $x: 0 \to 1$

$z = t^3$ $dz = 3t^2\,dt$ $y: 0 \to 1$

$z: 0 \to 1$

$\therefore \vec{F}\cdot d\bar{r} = (3t^2 + 6t^2)\,dt - 14t^2t^3\, 2t\,dt + 20\,t\,t^6\, 3t^2\,dt$

$\Rightarrow 9t^2\,dt - 28t^6\,dt + 60t^9\,dt \;;\; 0 \le t \le 1$

$$\int_{C} \ldots (\ldots + 60t^{9})\,dt$$
$t=0$

$$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10} \right]_0^1$$

$$\Rightarrow [3t^3 - 4t^7 + 6t^{10}]_0^1$$

$$\Rightarrow (3 - 4 + 6) - 0$$

$$\int_C \bar{F}\cdot d\bar{r} \Rightarrow 5$$

3) Evaluate $\int (5xy - 6x^2)\,dx + (2y - 4x)\,dy$ over the curve $y = x^3$ in the xy-plane from the point $(1,1)$ to $(2,8)$

$\bar{F} = (5xy - 6x^2)i + (2y-4x)j$

Sol^n: Let $\bar{r} = xi + yj + zk$ $\quad d\bar{r} = dxi + dyj + dzk$.

$\bar{F}\cdot d\bar{r} = [(5xy - 6x^2)i + (2y-4x)j]\,dxi + dyj + dzk$.

$$\Rightarrow (5xy - 6x^2)\,dx + (2y-4x)\,dy$$

But given that $y = x^3$

$dy = 3x^2 dx$

$\therefore \vec{F}\cdot d\bar{r} \Rightarrow (5x\,x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2 dx$.

$$\Rightarrow (5x^4 - 6x^2)\,dx + (6x^5 - 12x^3)\,dx$$

$$= (5x^4 - 6x^2 - 12x^3 + 6x^5)\,dx$$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_{x=1}^{2} (5x^4 - 6x^2 - 12x^3 + 6x^5)\,dx$$

$$= \left[\frac{5x^5}{5} - \frac{6x^3}{3} - \frac{12x^4}{4} + \frac{6x^6}{6} \right]_1^2$$

$$\Rightarrow [x^5 - 2x^3 - 3x^4 + x^6]_1^2$$

$$\Rightarrow 2^5 - 2^4 - 3(2)^4 + 2^6 - [1 - 2 - 3 + 1]$$

$$\Rightarrow 32 - 16 - 48 + 64 + 3$$

$$= 35$$

4) Find the total work done by the force represented by $\bar{F} = 3xyi - yj + 2zk$ in moving a particle round the circle $x^2 + y^2 = 4$

Sol^n: Total work done $W = \int_C \bar{F}\cdot d\bar{r}$

$x^2 + y^2 = 4$ can be represented in the parametric form of circle

$x = r\cos\theta$, $y = r\sin\theta$ ie $x^2 + y^2 = 2^2$ $\quad$ 0 to 2π "

$\boxed{x = 2\cos\theta}$ $\quad \boxed{y = 2\sin\theta}$ But in XY plane $z = 0$ $\quad dz = 0$

$dx = -2\sin\theta\, d\theta$, $\quad dy = 2\cos\theta\, d\theta$

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$z = 0 \quad ; \quad 0 \le \theta \le 2\pi$

$W = \int_C \bar{F} \cdot d\bar{r}$

$\Rightarrow \int_{\theta=0}^{2\pi} 3xy\,dx - y\,dy + 2zx\,dz$

$\Rightarrow \int_{\theta=0}^{2\pi} 3(2\cos\theta\, 2\sin\theta)(-2\sin\theta)d\theta - 2\sin\theta\, 2\cos\theta\, d\theta + 2(0)$

$= \int_{\theta=0}^{2\pi} -24(\sin^2\theta\cos\theta)d\theta - 4\sin\theta\cos\theta\, d\theta$

$= \int_{\theta=0}^{2\pi} -24[\sin^2\theta\cos\theta] - 2\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta$

$\Rightarrow -24\int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta - 2\int_0^{2\pi} \sin 2\theta\, d\theta$

$\Rightarrow -24\int_0^{2\pi} t^2\, dt - 2\int_0^{2\pi} \sin 2\theta\, d\theta$

Put $\sin\theta = t$
$\cos\theta\, d\theta = dt$
If $\theta = 0,\ t = 0$
$\theta = 2\pi,\ t = 0$

$\Rightarrow -24\left[\frac{t^3}{3}\right]_0^{2\pi} - 2\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$

$\Rightarrow -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} - 2\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$

$\sin 2\pi = 0$
$\sin 0 = 0$
$\cos 0 = 1$
$\cos 4\pi = 1$

$\Rightarrow -\frac{24}{3}\left[(\sin 2\pi)^3 - (\sin 0)^3\right] + \frac{2}{2}\left(\cos 4\pi - \cos 0)\right]$

$= 0 - 2(1 - 1)$

$= 0$ //

⑤. Find the Work done by the force $\vec{F} = (2y - x^2)i + 6yzj - 8xz^2k$ from the point $(0,0,0)$ to the Point $(1,1,1)$ along the straight line joining these points.

Solⁿ:→ Total Work done $W = \int_C \bar{F} \cdot d\bar{r}$

$\vec{F} = (2y - x^2)i + 6yzj - 8xz^2k$

Along the straight line $(0,0,0)$ to $(1,1,1)$

$\bar{r} = xi + yj + zk \qquad d\bar{r} = dxi + dyj + dzk$

$\bar{F} \cdot d\bar{r} = [(2y - x^2)i + 6yzj - 8xz^2k][dxi + dyj + dzk]$

$\bar{F} \cdot d\bar{r} \Rightarrow (2y - x^2)dx + 6yz\,dy - 8xz^2\,dz$

The straight line eqⁿ from $(0,0,0)$ to $(1,1,1)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Modu

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\Rightarrow x = t, \quad y = t, \quad z = t \quad \text{Where } t \text{ from } 0 \text{ to } 1$$

$$dx = dt \quad dy = dt \quad dz = dt$$

$$\bar{F}\cdot d\bar{r} = (2y - x^2)\,dx + 6yz\,dy - 8xz^2\,dz$$

$$\Rightarrow (2t - t^2)\,dt + 6tt\,dt - 8tt^2\,dt$$

$$= 2t\,dt - t^2\,dt + 6t^2\,dt - 8t^3\,dt$$

$$\int_C \bar{F}d\bar{r} \Rightarrow \int_C 2t\,dt + 5t^2\,dt - 8t^3\,dt.$$

$$\int_{t=0}^{1} (2t + 5t^2 - 8t^3)\,dt$$

$$\left[\frac{2t^2}{2} + \frac{5t^3}{3} - \frac{8t^4}{4}\right]_0^1 \Rightarrow 1 + \frac{5}{3} - 2 \Rightarrow \frac{5}{3} - 1 \Rightarrow \frac{2}{3}$$

6) Find the workdone in moving a particle in the force field $\vec{F} = 3x^2 i + (2xz - y) j + zk$ along the straight line from $(0,0,0)$ $(2,1,3)$

Soln: Total workdone $W = \int_C \bar{F}\cdot d\bar{r}$

Given, $\vec{F} = 3x^2 i + (2xz - y) j + zk$

$\bar{r} = xi + yj + zk$ $\qquad d\bar{r} = dxi + dyj + dzk$

$$\bar{F}\cdot d\bar{r} \Rightarrow [3x^2 i + (2xz - y) j + zk][dxi + dyj + dzk]$$

$$\Rightarrow 3x^2 dx + (2xz - y)\,dy + z\,dz$$

The straight line from $(0,0,0)$ to $(2,1,3)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\Rightarrow \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$= \frac{x}{2} = t, \quad \frac{y}{1} = t, \quad \frac{z}{3} = t$$

$$= x = 2t, \quad y = t, \quad z = 3t$$

$dx = 2dt, \quad dy = dt, \quad dz = 3dt$ Where,

$0 \leq t \leq 1$

$$\int_C \bar{F}\cdot d\bar{r} = 3(2t)^2\, 2dt + (2(2t)(3t) - t)\,dt + 3t\,(3\,dt)$$

$$\Rightarrow \int_{t=0}^{1} 24t^2\,dt + 12t^2\,dt - t\,dt + 9t\,dt$$

$$\Rightarrow \int_{t=0}^{1} 36t^2\,dt + 8t\,dt \Rightarrow \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$\Rightarrow 16$$

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from 0 to 1

Surface Integral :→ An integral of a vector point function evaluated over the surface is called surface integral.
Since the surface is involving only the two variables or (2 axis).

For a vector point function $\bar{F}$ over the surface S the surface integral is given by

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,ds$$

Where, $\bar{n}$ = unit normal vector or unit normal to the surface

Since the surface integral uses two variables. it is also calculated by

$$\iint_{xy\ Plane} \bar{F}\cdot\bar{n}\frac{dx\,dy}{|\bar{n}\cdot k|} \qquad \iint_{yz\ Plane} \bar{F}\cdot\bar{n}\frac{dy\,dz}{|\bar{n}\cdot i|}$$

$$\iint_{zx\ Plane} \bar{F}\cdot\bar{n}\frac{dz\,dx}{|\bar{n}\cdot j|}$$

Integrals Theorem

* Green's Theorem in a plane

Statement :→ Let R be a closed Region bounded by a closed curve C in XY plane and F_1 & F_2 be any two continuous & differentiable functions of x & y.

$$\int_C F_1dx + F_2dy = \iint_S\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

Ex ①. Verify Green's Theorem for $\int_C (xy+y^2)dx + x^2dy$ Where C is the closed curve C of the region bounded by $y = x$ and $y = x^2$

Solⁿ:→

Given $\int_C (xy+y^2)dx + x^2dy$

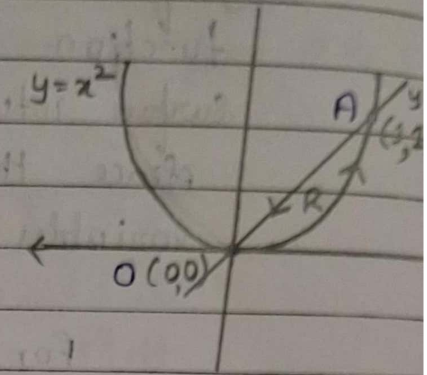
By green's Theorem We know that

$$\Rightarrow \int_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy \quad \text{---} (1).$$

Taking RHS.

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

R is the region bounded by $y = x^2$ & $y = x$



$\Rightarrow$ $F_1 = (xy + y^2)$ & $F_2 = x^2$

$\Rightarrow$ $\frac{\partial F_1}{\partial y} = (x + 2y)$ $\quad \frac{\partial F_2}{\partial x} = 2x$

$$\Rightarrow \iint_S (2x + x - 2y)\,dx\,dy$$

$$\Rightarrow \iint_S (x - 2y)\,dx\,dy$$

Given
$y = x$ & $y = x^2$
$x^2 = x$
$(x^2 - x) = 0$
$x(x-1) = 0$
$x = 0,\ x = 1$
At, $x = 0$, $y = 0$
At, $x = 1$, $y = 1$
$\therefore$ The Point of interset is $(0,0)$ $(1,1)$

$$\int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{1} \left[xy - \frac{2y^2}{2} \right]_{x^2}^{x} dx$$

$$\int_0^1 (x^2 - x^2 - x^3 + x^4)\,dx$$

$$\int_0^1 (x^4 - x^3)\,dx \Rightarrow \left. \frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} \Rightarrow -\frac{1}{20} \quad \text{RHS.}$$

LHS $\quad \int_C (xy + y^2)dx + x^2 dy = \int_{OA} + \int_{AO}$

Along OA curve
$y = x^2$
$dy = 2x\,dx$
Along AO curve
$y = x$
$dy = dx$

$$\int_{OA} (x x^2 + x^4)dx\,dx + x^2\,2x\,dx + \int_{AO} (x^2 + x^2)dx + x^2 dx$$

$$\Rightarrow \int_{x=0}^{1} (x^3 + x^4 + 2x^3)\,dx + \int_{x=1}^{0} 3x^2\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5} + \cancel{2x^4} \right]_0^1 + \left[\frac{3x^3}{3} \right]_1^0$$

Note :→ In double integral I Limit must be constant Limit second must be variable Limit.

$$\Rightarrow \frac{3}{4} + \frac{1}{5} + \cancel{} - 1$$

$$\Rightarrow \frac{15 + 4}{20} - 1$$

$$= \frac{19}{20} - 1 \Rightarrow -\frac{1}{20} \quad \text{--- LHS.}$$

$\therefore$ RHS = LHS $\quad$ Hence green's theorem is verified.

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3) Evaluate $\int xy\,dx + xy^2\,dy$ by Green's theorem Where ~~C~~ C is the square in the xy Plane with the vertices (1,0) (−1,0) (0,1) (0,−1)

Sol^n :→ By green's Theorem,

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \int_C F_1\,dx + F_2\,dy \quad \text{---}(1)$$

$\because F_1 = xy$ $\qquad F_2 = xy^2$

$\frac{\partial F_1}{\partial y} = x$ $\qquad \frac{\partial F_2}{\partial x} = y^2$

Square can be described by $-1 \le x \le 1$ and $-1 \le y \le 1$

RHS $\int_{x=-1}^{1}\int_{y=-1}^{1} (y^2 - x)\,dy\,dx$

$$\int_{-1}^{1} \left[\frac{y^3}{3}\right]_{-1}^{1} - [xy]_{-1}^{1}\,dx$$

$$\int_{-1}^{1} \left[\frac{1}{3} + \frac{1}{3}\right] - [x + x]\,dx$$

$$\int_{-1}^{1} \left(\frac{2}{3} - 2x\right) dx \Rightarrow \left[\frac{2}{3}x - \frac{2x^2}{2}\right]_{-1}^{1}$$

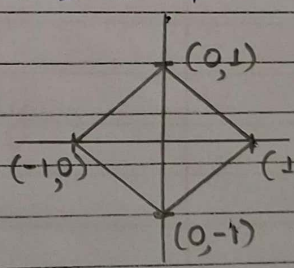
$$= \frac{2}{3} - 1 + \frac{2}{3} + 1 \Rightarrow 4/3 \text{ ------- RHS}$$

LHS $\Rightarrow \int_C (xy)\,dx + xy^2\,dy$

i) From (1,0) to (−1,0) :→ $\qquad x \to 1$ to -1

$y = 0 \quad \therefore dy = 0$

$$I_1 = \int_{x=1}^{-1} [x \cdot (0)\,dx + x(0)(0)]$$



$I_1 = 0$

I_2 = from (−1,0) to (0,1)

$x \to -1$, $dx = 0$ $\quad y \to 0$ to 1

$$I_2 = \int_{y=0}^{1} (-1)y(0) + (-1)y^2\,dy$$

$$\int_{y=0}^{1} -y^2\,dy \Rightarrow -\left[y^3/3\right]_0^1 \Rightarrow -\frac{1}{3}$$

$I_3 \Rightarrow (0,1)\ (0,-1)$

$x = 0 \qquad y \to 1 \text{ to } -1$

$dx = 0$

$$\int_{y=1}^{-1} 0(y)(0) + (0)\, y^2\, dy$$

$I_3 = 0$

$I_4 \Rightarrow \overset{x_1\ y_1}{(0,-1)} \to \overset{x_2\ y_2}{(1,0)}$

slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1}{1}$

$x : 0$ to 1

eqn of 2 point

$y - y_1 = m(x - x_1)$

$y - (-1) = 1(x - 0)$

$y + 1 = x$

$y = x - 1$

$dy = dx$

$$\int_{x=0}^{1} xy\, dx + xy^2\, dy$$

$$\int_0^1 x(x-1)\, dx + x(x-1)^2\, dx$$

$$= \int_0^1 x^2 - x + x[x^2 - 2x + 1]\, dx$$

$$\int_0^1 (x^2 - \cancel{x} + x^3 - 2x^2 + \cancel{x})\, dx$$

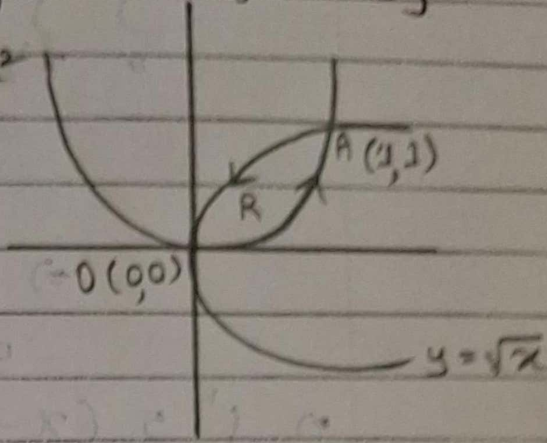
$$\int_0^1 (x^3 - x^2)\, dx \Rightarrow \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} \Rightarrow \frac{-1}{12}$$

$I = I_1 + I_2 + I_3 + I_4$

$= 0 - \dfrac{1}{3} + 0 - \dfrac{1}{12} \Rightarrow \dfrac{-4-1}{12} \Rightarrow -\dfrac{5}{12}$

Green's Theorem is not verified.

verify Green's Theorem in a plane for $\int_C (3x^2-8y^2)dx + (4y-6xy)dy$ Where C is the boundary of the region enclosed by $y=\sqrt{x}$ and $y=x^2$



Sol:→

We shall find Point of intersection of the Parabolas $y=\sqrt{x}$ & $y=x^2$

i.e. $y^2 = x$ & $y = x^2$

$x^4 = x$

$(x^4 - x) = 0$

$x(x^3-1) = 0$

$x=0, \quad x=1$

$(0,0) \; (1,1)$

$\int (3x^2-8y^2)dx + (4y-6xy)dy = \int_{OA} + \int_{AO}$

LHS Along OA Line :→

$y = x^2 \qquad dy = 2x\,dx$

$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$\int (3x^2-8x^4)\,dx + (4x^2 - 6x\cdot x^2)\,2x\,dx$

$\int [3x^2 - 8x^4 + 8x^3 - 12x^4]\,dx$

$\left[3\frac{x^3}{3} + 8\frac{x^4}{4} - 20\frac{x^5}{5}\right]_0^1$

$[1+2-4] \Rightarrow -1$

Along AO Line :→

$y = \sqrt{x}$ or $y^2 = x$

$2y\,dy = dx$

$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$\int_C (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^2y)\,dy$

$\int_{y=1}^{0} [6y^5 - 16y^3 + 4y - 6y^3]\,dy$

$\left[\frac{6y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_1^0$

$-\left[1 - \frac{11}{2} + 2\right] \Rightarrow -\left[3 - \frac{11}{2}\right] \Rightarrow \frac{5}{2}$

$\int_{OA} + \int_{AO} \Rightarrow -1 + \frac{5}{2} \Rightarrow 3/2$ — — LHS

RHS. $\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dx\,dy$

$\int_0^1\int_{x^2}^{\sqrt{x}} \frac{\partial}{\partial x}(4y^2-6yx) - \frac{\partial}{\partial y}(3x^2-8y^2)\,]\,dx\,dy$

$\int_0^1\int_{x^2}^{\sqrt{x}}(-6y+16y)\,dy\,dx$

$\Rightarrow \int_0^1\int_{x^2}^{\sqrt{x}} 10\,y\;dx.$

$\Rightarrow \int_0^1 10\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}}dx \Rightarrow \int_0^1 5[(\sqrt{x})^2-x^4]\,dx$

$\Rightarrow \int_0^1 5(x-x^4)dx \Rightarrow \int_0^1 5x-5x^4\,dx.$

$\Rightarrow \left.5\frac{x^2}{2}-5\frac{x^5}{5}\right]_0^1 \Rightarrow \frac{5}{2}-1 \Rightarrow \frac{3}{2}$ RHS

$\therefore$ LHS = RHS   Hence green's theorem is verified.

Stokes Theorem :→
Statement :→ Let S be the open surface bounded by a closed curve C. if $\bar{F}$ is a continuous and differentiable vector Point function then

$\int_C F\cdot d\bar{r} = \iint_S \text{curl } F\cdot n\,ds$ or $\iint_S (\nabla\times\bar{F})\,n\,ds$

here $\bar{r}$ is Position vector which is nothing but $xi+yj+zk$
& $\bar{n}$ is unit vector or unit normal vector
⇒ vector $\perp^{er}$ to the Plane.

Ex 1) Verify Stokes theorem for $F=(2x-y)i-yz^2j-y^2zk$ over the upper half surface of the sphere $x^2+y^2+z^2=1$

Solⁿ :→ Given $\bar{F}=(2x-y)i-yz^2j-y^2zk$

$d\bar{r} = dx\,i+dy\,j+dz\,k.$

$\int_C F\cdot d\bar{r} = \int_C (2x-y)dx-(yz^2)dy-(y^2z)k$

By stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times F)\hat{n}ds$ --- ①

$\therefore$ C is the boundary of upper half sphere which is circle whose eqⁿ is $x^2+y^2=1$, $z=0$ if Parametric eqⁿ of circle is

$x=a\cos\theta$, $dx=-a\sin\theta d\theta$ $y=a\sin\theta$ $dy=a\cos\theta d\theta$

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$z = 0$ & $0 \le \theta \le 2\pi$ $\quad dz = 0.$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2x+y)\,dx = \int_C (2a\cos\theta - a\sin\theta) - a\sin\theta\, d\theta$$

$$\Rightarrow \int_0^{2\pi} (-2\cos\theta\sin\theta + \sin^2\theta)\,d\theta = \int_0^{2\pi}\left[-\sin 2\theta + \tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$\Rightarrow \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$\Rightarrow \left[\frac{\cos 4\pi}{2} + \frac{1}{2}(2\pi - 0)\right] - \left[\frac{1}{2} + 0\right] \qquad \cos 4\pi = 1,\ \sin 2\pi = 0$$

$$= \frac{1}{2} + \pi - \frac{1}{2} = \pi \text{ --- LHS --- } (1).$$

- $\iint \text{curl } F\cdot n\, ds$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix}$$

$$i[-2yz + 2yz] - j[0-0] + k[0+1]$$

$$= k$$

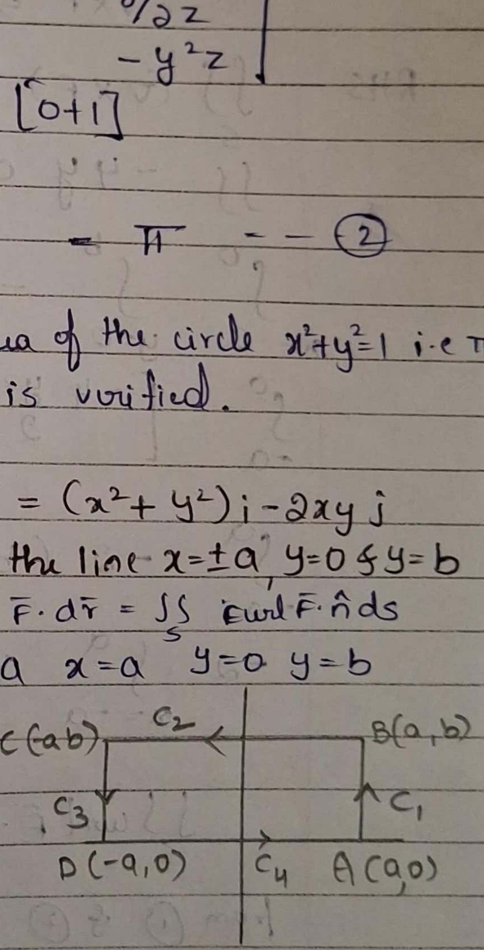
$$\iint_S \text{curl}\bar{F}\cdot\bar{n}\,ds = \iint dx\,dy = \pi \text{ --- } (2)$$

Since $\iint dx\,dy$ represents the area of the circle $x^2+y^2=1$ i.e π
Thus from (1) & (2) Stokes theorem is verified.

2) Verify Stokes theorem for $\bar{F} = (x^2+y^2)i - 2xy\,j$ taken round the rectangle bounded by the line $x = \pm a$, $y = 0$ & $y = b$

Soln:→ Given $\quad$ By Stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$

$\vec{F} = (x^2+y^2)i - 2xy\,j$, $\quad x=-a \quad x=a \quad y=0 \quad y=b$



$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $x = a$, $dx = 0$
y varies from 0 to b

Along BC, $y = b$, $dy = 0$
x varies from a to $-a$

Along CD, $x = -a$ $\quad dx = 0$
y varies from b to 0

Along DA, $y = 0$ $\quad dy = 0$
x varies from $-a$ to a.

$\bar{F}\cdot d\bar{r} = [(x^2+y^2)i - 2xyj][dxi+dyj+dzk]$

$\Rightarrow (x^2+y^2)\,dx - 2xy\,dy$

$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} [0-2ay]\,dy + \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy + \int_{x=a}^{a} x^2\,dx$

$\Rightarrow -2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^{-a} + 2a\left[\frac{y^2}{2}\right]_b^0 + \left[\frac{x^3}{3}\right]_{-a}^{a}$

$= -ab^2 + \left[\left(\frac{-a^3}{3}+b^2(-a)\right) - \left(\frac{a^3}{3}+b^2a\right)\right] + a[0-b^2] + \frac{1}{3}[a^3+a^3]$

$= -ab^2 + \left[-\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right] - ab + \frac{2a^3}{3}$

$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$

$\int_C \bar{F}\cdot d\bar{r} = -4ab^2$ ------------ (1).

$\nabla\times F = \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix} \Rightarrow k[-2y-2y]$

$\nabla\times F \Rightarrow -4yk.$ & $\hat{n} = k.$

RHS $\iint_S (\nabla\times F)\,\bar{n}\cdot ds.$

$\iint -4y\,dx\,dy$

$\int_{x=-a}^{0}\int_{y=0}^{b} -4y\,dy\,dx$

$\int_{-a}^{0}\left[-4\frac{y^2}{2}\right]_0^b dx$

$\int_{-a}^{a} -2(b^2-0)\,dx$

$-2\int_{x=-a}^{a} b^2\,dx. \Rightarrow -2b^2[x]_{-a}^{a}$

$\Rightarrow -2b^2(a+a)$

$\iint_S \text{curl } \vec{F}\cdot\hat{n}\,ds \Rightarrow -4ab^2$ -- 2 -- RHS

from (1) & (2) LHS = RHS Hence stokes theorem is verified.

HW 106 (1). verify stokes theorem for $\vec{F} = y^2i + x^2j - (x+z)k$ and C is the boundary of the triangle with vertices at (000)(100)&(110) Ans:→ 1/3

(2). verify stokes theorem for the vector $\vec{F} = (x^2+y^2)i - 2xy\,j$ taken round the triangle bounded by x=a, x=a· y=0 y=b Ans:→ $-2ab^2$