Exact Diff Eqⁿs :→

A necessary and the sufficient condition for the diff eqⁿ $M(x,y)\,dx + N(x,y)\,dy = 0$ to be an exact Eqⁿ is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further the solⁿ of the exact Eqⁿ is given by

$$\int M\cdot dx + \int N(y)\cdot dy = C$$

where in the first term we integrate $M(x,y)$ w.r.to x keeping y fixed, $N(y)$ indicate the terms in N without x (not containing x)

Problems :→

1) solve $(2x+y+1)\,dx + (x+2y+1)\,dy = 0$

solⁿ :→ Let $M(x,y)\,dx + N(x,y)\,dy = 0$

Let $M = 2x+y+1$ and $N = x+2y+1$

$\therefore \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given Eqⁿ is Exact

The solⁿ is $\int M\,dx + \int N(y)\,dy = C$

(where $N(y)$ denotes terms in N not containing x)

i.e $\int (2x+y+1)\,dx + \int (2y+1)\,dy = C$

i.e $\frac{\cancel{2}x^2}{\cancel{2}} + xy + x + \frac{\cancel{2}y^2}{\cancel{2}} + y = C$

$x^2 + xy + x + y^2 + y = C$

$x^2 + y^2 + xy + x + y = C$ is Required solⁿ

2) solve $(y^3 - 3x^2y)\,dx - (x^3 - 3xy^2)\,dy = 0$

solⁿ: $M(x,y)\,dx + N(x,y)\,dy = 0$

Let $M = y^3 - 3x^2y$ and $N = -x^3 + 3xy^2$

$\frac{\partial M}{\partial y} = 3y^2 - 3x^2$ $\qquad \frac{\partial N}{\partial x} = -3x^2 + 3y^2$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given Eqⁿ is Exact

The solⁿ is $\int M\,dx + \int N(y)\,dy = C$

i.e $\int (y^3 - 3x^2y)\,dx + \int 0\,dy = C$

$y^3\cdot x - \frac{3x^3}{3}\cdot y = C \Rightarrow xy^3 - x^3y = C$

Thus $xy^3 - x^3y = C$ is the required solⁿ

3) solve $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

solⁿ: Let $M = 5x^4 + 3x^2y^2 - 2xy^3$

$N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ & $\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given Eqⁿ is exact

The solⁿ is $\int M\cdot dx + \int N(y)\,dy = C$

i.e $\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = C$

$\Rightarrow \frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{3}x^3y^2}{\cancel{3}} - \frac{\cancel{2}x^2y^3}{\cancel{2}} - \frac{\cancel{5}y^5}{\cancel{5}} = C$

$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$ is required solⁿ.

4) solve : $\frac{dy}{dx} + \frac{x+3y-4}{3x+9y-2} = 0$

The given eqⁿ is Equivalent to the form

$(x+3y-4)\,dx + (3x+9y-2)\,dy = 0$

$M = x+3y-4$ & $N = 3x+9y-2$

$\frac{\partial M}{\partial y} = 3$ $\quad \frac{\partial N}{\partial x} = 3$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given eqⁿ is exact

The solution is $\int M\cdot dx + \int N(y)\,dy = C$

i.e $\int (x+3y-4)\,dx + \int (9y-2)\,dy = C$

i.e $\frac{x^2}{2} + 3xy - 4x + \frac{9y^2}{2} - 2y = 0$

$x^2 + 6xy - 8x + 9y^2 - 4y = 2C$ is the required solⁿ.

5) solve $[y(1+1/x) + \cos y]\,dx + [x + \log x - x\sin y]\,dy = 0$

solⁿ: let $M = y\left(1+\frac{1}{x}\right) + \cos y$ and $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right) - \sin y$ $\quad \frac{\partial N}{\partial x} = \left(1+\frac{1}{x}\right) - \sin y$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given eqⁿ is exact

The solⁿ is $\int M\cdot dx + \int N(dy)\,dy = C$

i.e $\int [y(1+\frac{1}{x}) + \cos y]\,dx + \int 0\cdot dy = C$

Thus $y(x+\log x) + x\cos y = C$ is the required solⁿ

6) solve $[\cos x\cdot\tan y + \cos(x+y)]\,dx + [\sin x\cdot\sec^2 y + \cos(x+y)]\,dy = 0$

solⁿ :⇒ let $M = \cos x.\tan y + \cos(x+y)$

$N = \sin x\cdot\sec^2 y + \cos(x+y)$

∴ $\frac{\partial M}{\partial y} = \cos x\cdot\sec^2 y - \sin(x+y)$.

$\frac{\partial N}{\partial x} = \cos x\cdot\sec^2 y - \sin(x+y)$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given eqⁿ is Exact

The solⁿ is $\int M\cdot dx + \int N(y)\cdot dy = C$

i.e $\int [\cos x.\tan y + \cos(x+y)]\,dx + \int 0\,dy = C$

Thus. $\sin x.\tan y + \sin(x+y) = C$ is the required solⁿ.

7) Solve : $ye^{xy}.dx + (x.e^{xy} + 2y)\,dy = 0$

solⁿ :⇒ let $M = y.e^{xy}$. $\quad N = x.e^{xy} + 2y$

$\frac{\partial M}{\partial y} = y.e^{xy}.x + e^{xy}$. $\quad \frac{\partial N}{\partial x} = xe^{xy}.y + e^{xy}$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the given eqⁿ is exact

solⁿ is given by $\int M\,dx + \int N(y)\,dy = C$

i.e $\int y.e^{xy}.dx + \int 2y\,dy = C$

$y.\frac{e^{xy}}{y} + y^2 = C$

$e^{xy} + y^2 = C$ is the required solⁿ.

8) S.T the following Eq^n is an exact differential Eq^n & hence solve the same.

$$(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$$

Sol^n: let $M = 1+e^{x/y}$ and $N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$. $\quad \frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$

i.e $\frac{\partial M}{\partial y} = -\frac{x\cdot e^{x/y}}{y^2}$ $\quad \frac{\partial N}{\partial x} = \frac{-1}{y}e^{x/y} + \frac{1}{y}e^{x/y} - \frac{x}{y^2}e^{x/y}$

$$= -\frac{x e^{x/y}}{y^2}$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given Eq^n is Exact

The sol^n is $\int M\,dx + \int N(y)\,dy = c$

i.e $\int (1+e^{x/y})\,dx + \int 0\cdot dy = c$

i.e $x + \frac{e^{x/y}}{(1/y)} = c$

Thus $x + y\cdot e^{x/y} = c$ is required sol^n.

Eq^{n}s reducible to the exact form.

Sometimes the given differential Eq^n which is not an exact Eq^n can be transformed into an exact Eq^n by multiplying with some fun^n known as the Integrating factor.

The procedure to find such a factor is as follows.

Suppose that, for the Eq^n $M\,dx + N\,dy = 0$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then we take their difference

The difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ should be close to the Expression of M or N.

If it is so, then we compute

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \quad \text{or} \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$

then $e^{\int f(x)\,dx}$ or $e^{-\int g(y)\,dy}$ is an integrating factor.

As already stated the multiplication of this factor into the Eq^n $M\,dx + N\,dy = 0$ will make the Eq^n an exact one and we solve this modified Eq^n being an exact Eq^n.

$e^{\log x} = x$ and $e^{n\log x} = x^n$

1) solve $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0$

Sol^n: let $M = 4xy + 3y^2 - x$ $\quad N = x(x+2y) = x^2 + 2xy$

$\frac{\partial M}{\partial y} = 4x + 6y$ $\quad \frac{\partial N}{\partial x} = 2x + 2y$

(The Eq^n is not exact)

Consider, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y = 2(x+2y)$

(close to N)

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$

Hence, $e^{\int f(x)\,dx}$ is an integrating factor i.e $e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

Multiplying the given Eq^n by x^2 we now have

$M = 4x^3y + 3x^2y^2 - x^3$ & $N = x^4 + 2x^3y$

$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y$ & $\frac{\partial N}{\partial x} = 4x^3 + 6x^2y$

Solution of Exact Eq^n is $\int M \cdot dx + \int N(y) dy = c$

i.e $\int (4x^3y + 3x^2y^2 - x^3) dx + \int 0 \, dy = c$

Thus $x^4y + x^3y^2 - \frac{x^4}{4} = c$ is required sol^n.

2) Solve $(x^2+y^2+x)dx + xy\,dy = 0$

sol^n: let $M = x^2+y^2+x$ & $N = xy$

$\frac{\partial M}{\partial y} = 2y$ & $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$ (near to N.)

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x)$

Hence IF $= e^{\int f(x)dx} = e^{\int 1/x \, dx} = e^{\log x} = x$

Multiplying the given Eq^n by x we now have,

$M = x^3 + xy^2 + x^2$ and $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$ $\frac{\partial N}{\partial x} = 2xy$

The sol^n is $\int M \cdot dx + \int N(y) dy = c$

i.e $\int (x^3 + xy^2 + x^2) dx + \int 0 \, dy = c$

Thus $\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$ is the required sol^n.

Solve

3) $y(2xy+1)dx - x\,dy = 0$

sol^n: let $M = y(2xy+1)$ and $N = -x$

$\frac{\partial M}{\partial y} = 4xy + 1$ and $\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy + 1 + 1 = 4xy + 2 \Rightarrow 2(2xy+1)$

(near to M)

Now, $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2(2xy+1)}{y(2xy+1)} = \frac{2}{y} = g(y)$

Hence I.F $= e^{-\int g(y)dy}$

$= e^{-\int 2/y \cdot dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

Multiplying the given Eq^n with $\frac{1}{y^2}$, we now have

$M = 2x + \frac{1}{y}$ and $N = \frac{-x}{y^2}$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2}$ $\frac{\partial N}{\partial x} = \frac{-1}{y^2}$

The sol^n is $\int M \cdot dx + \int N(y) \cdot dy = c$

i.e $\int \left(2x + \frac{1}{y}\right) dx + \int 0 \, dy = c$

Thus $x^2 + \frac{x}{y} = c$ is the required sol^n.

4) Solve $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$

sol^n: let $M = 8xy - 9y^2$ & $N = 2x^2 - 6xy$

$\frac{\partial M}{\partial y} = 8x - 18y$ and $\frac{\partial N}{\partial x} = 4x - 6y$

$\frac{\partial M}{\partial N} - \frac{\partial N}{\partial x} = 4x - 12y = 4(x - 3y)$ is near to $N = 2x(x-3y)$

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{4(x-3y)}{2x(x-3y)} = \frac{2}{x} = f(x)$

Hence, $I.F = e^{\int f(x)dx}$
$= e^{\int 2/x\, dx} = e^{2\log x} = x^2$

Multiplying the given Eq^n by x^2 we now have
$M = 8x^3y - 9x^2y^2$ and $N = 2x^4 - 6x^3y$
$\frac{\partial M}{\partial y} = 8x^3 - 18x^2y$ and $\frac{\partial N}{\partial x} = 8x^3 - 18x^2y$

The sol^n is $\int M.dx + \int N(y)dy = C$

i.e $\int (8x^3y - 9x^2y^2)dx + \int 0\, dy = C$

Thus $2x^4y - 3x^3y^2 = c$ is required sol^n.

Bernoulli's Diff Eq^n :=>

A Diff Eq^n is said to be linear if the dependent variable and its derivative occurs in the first degree only and they are not multiplied together.

* Std form of a linear Eq^n and its solution

A diff Eq^n of the form
$\frac{dy}{dx} + Py = Q$ —— (1)
where P and Q are fun^{n}s of x only is called a linear Eq^n in 'y'

The solution can be ~~delivered~~ derived in the following form
$y e^{\int P dx} = \int Q . e^{\int P dx} . dx + C$

An Eq^n of the form : $\frac{dx}{dy} + Px = Q$ where P and Q are fun^n of y only is called a linear Eq^n in x. The solution can simply be written by interchanging the role of x & y given is the sol^n for the linear Eq^n in x
i.e $x.e^{\int P dy} = \int Q . e^{\int P dy} . dy + C$

Note :-
1) When the DE is in the linear form (y or x) we assume the solution in the appropriate form and obtain the sol^n identifying P & Q
2) The terms $e^{\int P dx}$ and $e^{\int P dy}$ are called Integrating factors.

Ex:- 1) solve $\frac{dy}{dx} + y\cot x = \cos x$

Sol^n: $\frac{dy}{dx} + y\cot x = \cos x$ is of the form
$\frac{dy}{dx} + Py = Q$, where $P = \cot x$ & $Q = \cos x$
$\therefore e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log(\sin x)} = \sin x$
The solution is
$y.e^{\int P dx} = \int Q . e^{\int P dx} . dx + C$
i.e $y \sin x = \int \cos x . \sin x . dx + C$

$\int uv\,dx = u\int v\,dx - \int (u'\int v\,dx)\,dx$

i.e $y \sin x = \int \frac{\sin 2x}{2}\,dx + c$

Thus $y \sin x = \frac{-\cos 2x}{4} + c$ is the required soln.

Ex 2: solve $\sqrt{1-y^2}\,.dx = (\sin^{-1}y - x)\,dy$.

soln: It is not possible to solve if the DE is written in the form.

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{[\sin^{-1}\{y-x\}]}$$

$$\frac{dx}{dy} = \frac{\sin^{-1}\{y-x\}}{\sqrt{1-y^2}}$$

or $\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$ and this is of the form

$\frac{dx}{dy} + px = Q$ where $p = \frac{1}{\sqrt{1-y^2}}$ and $Q = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$

$\therefore e^{\int p\,dy} = e^{\sin^{-1}y}$

The soln is, $x e^{\int p\,dy} = \int Q . e^{\int p\,dy}\,dy + c$

i.e $x . e^{\sin^{-1}y} = \int \frac{\sin^{-1}y}{\sqrt{1-y^2}} . e^{\sin^{-1}y} . dy + c$

put $\sin^{-1}y = t$.

$\therefore \frac{1}{\sqrt{1-y^2}}\,dy = dt$

soln: $x . e^{\sin^{-1}y} = \int t . e^t . dt + c$

i.e $x . e^{\sin^{-1}y} = t . e^t - e^t + c$, on integration by parts

Thus.

$x . e^{\sin^{-1}y} = e^{\sin^{-1}y}(\sin^{-1}y - 1) + c$ is the required soln.

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The DE of the form : $\frac{dy}{dx} + Py = Qy^n$

where P and Q are funcns of x is called as Bernoulli's differential Eqn in y.

we first divide the eqn throughout by y^n to obtain.

$$\frac{1}{y^n}\frac{dy}{dx} + p . y^{1-n} = Q \quad \text{———} \quad (1)$$

put $y^{1-n} = t$. $\therefore (1-n)y^{-n} . \frac{dy}{dx} = \frac{dt}{dx}$

or $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}$

Hence (1) becomes, $\frac{1}{(1-n)}\frac{dt}{dx} + Pt = Q$

or $\frac{dt}{dx} + (1-n)P . t = (1-n)Q$ (which is a linear Eqn in t)

Similarly,

$\frac{dx}{dy} + px = Qx^n$ where P & Q are funcns of y is called Bernoulli's Eqn in x. we first divide by x^n & later put $x^{1-n} = t$. Diff w.r.to y to obtain a linear Eqn in $t = t(y)$

1) solve : $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

soln: This is Bernoulli's Eqn. Dividing the given Eqn by y^2 we have

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{yx} = x \quad \text{——} ①$$

put $\frac{1}{y} = t$, $\therefore \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence ① becomes, $\frac{-dt}{dx} + \frac{t}{x} = x$ or $\frac{dt}{dx} - \frac{t}{x} = -x$

This eqⁿ is a linear eqⁿ of the form $\frac{dt}{dx} + Pt = Q$ where

$P = \frac{-1}{x}$ and $Q = -x$

$e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

The solⁿ is. $t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$

i.e. $t \cdot \frac{1}{x} = \int -x \cdot \frac{1}{x}\, dx + C$

Thus $\frac{1}{xy} = -x + C$ is required solⁿ.

(02) solve : $\frac{dy}{dx} - y\tan x = \frac{\sin x \cdot \cos^2 x}{y^2}$

solⁿ: Multiplying the given eqⁿ by y^2 we have

$y^2 \cdot \frac{dy}{dx} - y^3 \cdot \tan x = \sin x \cdot \cos^2 x$ —— ①

put $y^3 = t$

$\therefore 3y^2 \cdot \frac{dy}{dx} = \frac{dt}{dx}$ or $y^2 \frac{dy}{dx} = \frac{1}{3}\frac{dt}{dx}$

Hence ① becomes, $\frac{1}{3}\frac{dt}{dx} - t\tan x = \sin x \cos^2 x$

or $\frac{dt}{dx} - 3\tan x \cdot t = 3\sin x \cdot \cos^2 x$

This eqⁿ is of the form, $\frac{dt}{dx} + Pt = Q$, where $P = -3\tan x$ and $Q = 3\sin x \cos^2 x$

$\therefore e^{\int P dx} = e^{\int -3\tan x\, dx}$

$= e^{-3\log(\sec x)}$

$= (\sec x)^{-3} = \cos^3 x$

The solution is, $t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$

i.e $t \cdot \cos^3 x = \int 3\sin x \cdot \cos^2 x \cdot \cos^3 x \cdot dx + C$

i.e $t\cos^3 x = 3\int \sin x \cdot \cos^5 x \cdot dx + C$

put $\cos x = u$ $\therefore -\sin x\, dx = du$

$\therefore t\cos^3 x = -3\int u^5\, du + C$

i.e $t \cdot \cos^3 x = \frac{-u^6}{2} + C$

Thus, $(y\cos x)^3 = \frac{-\cos^6 x}{2} + C$

is the required solⁿ.

3> solve $x^3 \cdot \frac{dy}{dx} - x^2 y = -y^4 \cos x$

solⁿ :⇒

3> Solve : $r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$

solⁿ: we have $\cos\theta \cdot \frac{dr}{d\theta} - r\sin\theta = -r^2$

Dividing by r^2 we get

$$\frac{\cos\theta}{r^2}\frac{dr}{d\theta} - \frac{1}{r}\sin\theta = -1 \quad \text{——} ①$$

put $\frac{1}{r} = y$ and Diff w.r.to θ

$\therefore \frac{-1}{r^2}\frac{dr}{d\theta} = \frac{dy}{d\theta}$ and hence ① becomes

$-\cos\theta \cdot \frac{dy}{d\theta} - y\sin\theta = -1$

or $\frac{dy}{d\theta} + \tan\theta \cdot y = \sec\theta$

This is of the form $\frac{dy}{d\theta} + Py = Q$ where

$P = P(\theta) = \tan\theta$ and $Q = Q(\theta) = \sec\theta$

Here $IF = e^{\int P d\theta}$

$= e^{\int \tan\theta \cdot d\theta}$

$= e^{\log(\sec\theta)}$

$= \sec\theta$

sol^n is given by

$y \cdot e^{\int P d\theta} = \int Q \cdot e^{\int P d\theta} \cdot d\theta + c$

i.e $y \cdot \sec\theta = \int \sec^2\theta \cdot d\theta + c$

i.e $y\sec\theta = \tan\theta + c$

Thus $\frac{\sec\theta}{r} = \tan\theta + c$ is required sol^n.

4) solve $y(2xy + e^x)dx - e^x dy = 0$

sol^n: The given eq^n can be written in the form

$\frac{dy}{dx} = \frac{y(2xy + e^x)}{e^x}$

$\frac{dy}{dx} - y = \frac{2xy^2}{e^x}$ Dividing by y^2 we get

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x}$ —— ①

put $\frac{1}{y} = t$ $\therefore \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence ① becomes

$-\frac{dt}{dx} - t = \frac{2x}{e^x}$

or $\frac{dt}{dx} + t = \frac{-2x}{e^x}$

The eq^n is of the form $\frac{dt}{dx} + Pt = Q$ where

$P = 1$ and $Q = -\frac{2x}{e^x}$

$IF = e^{\int P dx} = e^{\int 1 dx} = e^x$

The sol^n is $t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + c$

i.e $t \cdot e^x = \int -\frac{2x}{e^x} \cdot e^x \cdot dx + c$

$t \cdot e^x = -x^2 + c$

Thus $\frac{e^x}{y} + x^2 = c$ is required sol^n

5) solve $\frac{dy}{dx} - \frac{1}{2}\left(1 + \frac{1}{x}\right)y + \frac{3y^3}{x} = 0$

sol^n: $\frac{dy}{dx} - \frac{1}{2}\left(1 + \frac{1}{x}\right)y = -\frac{3y^3}{x}$

Divide by y^3

$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2}\left(1 + \frac{1}{x}\right)\frac{1}{y^2} = \frac{-3}{x}$ —— ①

put $\frac{1}{y^2} = t$ $\frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$

$$-\frac{1}{2}\frac{dt}{dx}-\frac{t}{2}\left(1+\frac{1}{x}\right)=-\frac{3}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{dt}{dx}+t\left(1+\frac{1}{x}\right)\right]=+\frac{3}{x}$$

$$\frac{dt}{dx}+t\left(1+\frac{1}{x}\right)=\frac{6}{x}$$

This Eq.^{n} is of the form, $\frac{dt}{dx}+Pt=Q$ where

$P=1+\frac{1}{x}$ and $Q=\frac{6}{x}$

$IF=e^{\int P dx}=e^{\int 1+\frac{1}{x}dx}=e^{x+\log x}=e^{x}.e^{\log x}=xe^{x}$

The sol.^{n} is

$t.e^{\int P dx}=\int Q.\,e^{\int P dx}\,dx+c$

$t.x.e^{x}=\int \frac{6}{\cancel{x}}.\cancel{x}.e^{x}.dx+c$

$t.x.e^{x}=\int 6e^{x}\,dx+c$

$t.x.e^{x}=6.e^{x}+C$ $\qquad t=\frac{1}{y^2}$

$\frac{x.e^{x}}{y^2}=6e^{x}+C$

6) solve $x^3.\frac{dy}{dx}-x^2y=-y^4\cos x$

sol.^{n}: Dividing $x^3.y^4$ we get

$$\frac{1}{y^4}\frac{dy}{dx}-\frac{1}{xy^3}=\frac{-\cos x}{x^3} \quad — ①$$

put, $\frac{1}{y^3}=t$ $\quad\therefore \frac{-3}{y^4}\frac{dy}{dx}=\frac{dt}{dx}$

$$\frac{1}{y^4}\frac{dy}{dx}=\frac{-1}{3}\frac{dt}{dx}$$

Hence ① becomes, $\frac{-1}{3}\frac{dt}{dx}-\frac{t}{x}=-\frac{\cos x}{x^3}$

i.e $\frac{dt}{dx}+3\frac{t}{x}=\frac{3\cos x}{x^3}$

This Eq.^{n} is a linear Eq.^{n} of the form $\frac{dt}{dx}+Pt=Q$ where

$P=\frac{3}{x}$ and $Q=\frac{3\cos x}{x^3}$

$IF=e^{\int P.dx}=e^{\int 3/x\,dx}=e^{3\log x}=x^3$

Sol.^{n} is given by

$t.e^{\int P dx}=\int Q.\,e^{\int P dx}\,dx+c$

$t.\,x^3=\int \frac{3\cos x}{\cancel{x^3}}.\cancel{x^3}dx+C$

$tx^3=3\int \cos x\,dx+c$

$tx^3=3\sin x+C$

$\frac{x^3}{y^3}=3\sin x+c$ is required sol.^{n}.

7) solve: $\frac{dy}{dx}=\frac{y}{x+\sqrt{xy}}$

$\Rightarrow$ Noting that RHS can't be simplified we consider $\frac{dx}{dy}=\frac{x+\sqrt{xy}}{y}$

$$\frac{dx}{dy}-\frac{x}{y}=\frac{\sqrt{x}}{\sqrt{y}}$$

we have $\frac{1}{\sqrt{x}}\frac{dx}{dy}-\frac{\sqrt{x}}{y}=\frac{1}{\sqrt{y}}$ $\quad — ①$

put $\sqrt{x}=t$ and Diff w.r.to y

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy}=\frac{dt}{dy} \quad \text{or} \quad \frac{1}{\sqrt{x}}\frac{dx}{dy}=2\frac{dt}{dy}$$

Hence ① becomes

$$2\cdot\frac{dt}{dy}-\frac{t}{y}=\frac{1}{\sqrt{y}} \quad \text{or} \quad \frac{dt}{dy}-\left(\frac{1}{2y}\right)t=\frac{1}{2\sqrt{y}}$$

This Eq^n is of the form $\frac{dt}{dy}+Pt=Q$ where

$$P=\frac{-1}{2y} \quad \& \quad Q=\frac{1}{2\sqrt{y}}$$

$$IF=e^{\int P dy}=e^{\int -\frac{1}{2y}dy}=e^{-\frac{1}{2}\log y}=e^{\log y^{-1/2}}=1/\sqrt{y}$$

The sol^n is

$$t\cdot e^{\int P dy}=\int Q\cdot e^{\int P dy}\cdot dy+c$$

i.e $\frac{t}{\sqrt{y}}=\int\frac{1}{2y}dy+c$

i.e $\frac{t}{\sqrt{y}}=\frac{1}{2}\log y+c$ or $\frac{t}{\sqrt{y}}=\log\sqrt{y}+c$

Thus $\sqrt{x/y}-\log\sqrt{y}=c$
is the required sol^n.

$f(x,y)=c$

Applications of D.E

Orthogonal trajectories

Problems

1) Find the orthogonal trajectories of the family of parabolas $y^2=4ax$.

sol^n: consider, $\frac{y^2}{x}=4a$ —— ①

(If the parameter is on one side of the Eq^n exclusively, then the same gets eliminated once we differentiate)

Now diff ① w.r.to x we have

$$\frac{x\cdot 2y\frac{dy}{dx}-y^2\cdot 1}{x^2}=0 \quad \text{or} \quad 2xy\cdot\frac{dy}{dx}-y^2=0.$$

i.e $2x\cdot\frac{dy}{dx}-y=0$, is the DE of the given family

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right)-y=0$$

or $2x\,dx+y\cdot dy=0$

$\Rightarrow \int 2x\cdot dx+\int y\,dy=c$

i.e $x^2+\frac{y^2}{2}=c$ or $2x^2+y^2=2c=k$ (say)

Thus $2x^2+y^2=k$ is required orthogonal trajectory

2) Find the orthogonal trajectories of the family of astroids $x^{2/3}+y^{2/3}=a^{2/3}$

Sol^n: Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Diff w.r to x we have

$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \cdot \frac{dy}{dx} = 0$

i.e $x^{-1/3} + y^{-1/3} \cdot \frac{dy}{dx} = 0$ is the DE of the given family

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0$ or $x^{-1/3} dy = y^{-1/3} dx$

i.e $y^{1/3} dy = x^{1/3} dx$ by seperating variables

$\Rightarrow \int y^{1/3} dy - \int x^{1/3} dx = c$

i.e $\frac{y^{4/3}}{(4/3)} - \frac{x^{4/3}}{(4/3)} = c$

or $x^{4/3} - y^{4/3} = \frac{-4c}{3} = k$ (say)

Thus $x^{4/3} - y^{4/3} = k$ is required orthogonal trajectory.

3) Find the orthogonal trajectories of the family $y^2 = cx^3$

Sol^n: we have, $\frac{y^2}{x^3} = c$

Diff w.r.to x we have

$\frac{x^3 \cdot 2y \cdot \frac{dy}{dx} - y^2 \cdot 3x^2}{x^6} = 0$ or

$2x^3 y \frac{dy}{dx} = 3x^2 y^2$

i.e $2x \cdot \frac{dy}{dx} = 3y$ Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have.

$2x\left(-\frac{dx}{dy}\right) = 3y$ or $-2x\,dx = 3y\,dy$

i.e $2x\,dx + 3y\,dy = 0 \Rightarrow \int 2x\,dx + \int 3y\,dy = c$

i.e $x^2 + \frac{3y^2}{2} = c$ or $2x^2 + 3y^2 = 2c = k$ (say)

Thus $2x^2 + 3y^2 = k$ is required orthogonal trajectory.

4) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is the parameter.

Sol^n: we have, $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ — (1)

Diff w.r.to x we have

$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0$ where $y_1 = \frac{dy}{dx}$

i.e $\frac{x}{a^2} = \frac{-yy_1}{b^2+\lambda}$ — (2)

Also from (1) $\frac{x^2}{a^2} - 1 = \frac{-y^2}{b^2+\lambda}$

or $\frac{x^2 - a^2}{a^2} = \frac{-y^2}{b^2+\lambda}$ — (3)

Now dividing (2) by (3) we get

$\frac{x}{x^2 - a^2} = \frac{yy_1}{y^2}$ or $\frac{x}{x^2-a^2} = \frac{y_1}{y}$

Replacing $y_1 = \frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\frac{x}{x^2-a^2} = \frac{1}{y}\left(-\frac{dx}{dy}\right)$$

i.e $y \cdot dy = \frac{-(x^2-a^2)}{x} dx$ by seperating variables

$\Rightarrow \int y\,dy = -\int x\,dx + a^2 \int \frac{dx}{x} + c$

i.e $\frac{y^2}{2} = -\frac{x^2}{2} + a^2 \log x + c$

Thus $x^2+y^2-2a^2\log x - b = 0$
where $b = 2c$, is required O.T

5) Show that the family of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Sol^n: Consider $y^2 = 4a(x+a)$ —— ①

Diff w.r.to x we have

$2y \cdot \frac{dy}{dx} = 4a$

$2yy_1 = 4a$

$\frac{yy_1}{2} = a$ where $y_1 = \frac{dy}{dx}$

Substituting the value of 'a' in ① we have

$y^2 = 2yy_1\left(x + \frac{yy_1}{2}\right)$

or $y = 2xy_1 + yy_1^2$ —— ②

This is the DE of the given family

Now replacing y_1 by $\frac{-1}{y_1}$ ② becomes

$$y = 2x\left(\frac{-1}{y_1}\right) + y\left(\frac{-1}{y_1}\right)^2$$

or $y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$

i.e $yy_1^2 + 2xy_1 = y$ —— ③

Eq^n ③ is the DE of the orthogonal family which is same as ② being the DE of the given family.

Thus family of parabolas $y^2 = 4a(x+a)$ is self orthogonal

6) Using the concept of orthogonal trajectories S.T the family of curves $r = a(\sin\theta + \cos\theta)$ and $r = b(\sin\theta - \cos\theta)$ intersect each other orthogonally.

Sol^n: we shall S.T the O.T of the first family $r = a(\sin\theta + \cos\theta)$ is the other family $r = b(\sin\theta - \cos\theta)$

Consider $r = a(\sin\theta + \cos\theta)$

$\Rightarrow \log r = \log a + \log(\sin\theta + \cos\theta)$

Diff w.r.to θ we have

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \cdot \frac{d\theta}{dr}$ we have

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \text{ or } -r \cdot \frac{d\theta}{dr} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \cdot d\theta = \frac{dr}{-r}$ by seperating variables

$$\Rightarrow \int \frac{dr}{r} + \int \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \cdot d\theta = c$$

i.e $\log r - \log(\cos\theta - \sin\theta) = c$

i.e $\log\left[\dfrac{r}{\cos\theta - \sin\theta}\right] = \log k$

$r = k(\cos\theta - \sin\theta)$ or $r = -k(\sin\theta - \cos\theta)$

and let $k = -b$

Thus we have $r = b(\sin\theta - \cos\theta)$ as required

7) Find the orthogonal trajectory of the family of curves $r^n = a^n \cos n\theta$

sol^n:- $r^n = a^n \cos n\theta$

$\Rightarrow n\log r = n\log a + \log(\cos n\theta)$

Diff w.r.to θ we get

$\dfrac{n}{r}\cdot\dfrac{dr}{d\theta} = \dfrac{-n\sin n\theta}{\cos n\theta}$ or $\dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we get

$\dfrac{1}{r}\left(-r^2\dfrac{d\theta}{dr}\right) = -\tan n\theta$

or $r\cdot\dfrac{d\theta}{dr} = \tan n\theta$

$\dfrac{d\theta}{\tan n\theta} = \dfrac{dr}{r}$

$-\displaystyle\int \dfrac{d\theta}{\tan n\theta} + \int \dfrac{dr}{r} = c$

or $\displaystyle\int \dfrac{dr}{r} - \int \cot n\theta\, d\theta = c$

$\log r - \dfrac{1}{n}\log(\sin n\theta) = c$

or $n\log r - \log(\sin n\theta) = nc$

i.e $\log\left[\dfrac{r^n}{\sin n\theta}\right] = \log k$ (say)

$r^n = k\sin n\theta$

Thus $r^n = k\sin n\theta$ is required orthogonal T.

Non-linear Differential Equations :-

If $y = f(x)$ we use the notation $\dfrac{dy}{dx} = p$ throughout this unit

The DE will involve $\dfrac{dy}{dx}$ in higher degree.

and $\dfrac{dy}{dx}$ is denoted by p. So the DE is of the form $f(x, y, p) = 0$ is called 1st order non-linear differential equation.

Such eq^{n}s can be solved by following methods

1) Equations solvable for p

2) Equation solvable for y

3) Eq^n solvable for x

Eq^{n}s solvable for p :-

A DE of 1st order & n^{th} degree is of the form

$A_0p^n + A_1p^{n-1} + A_2p^{n-2} + \dots + A_n = 0$ —— (1)

where $A_0, A_1, A_2, \dots, A_n$ are fun^n of x & y.

Suppose the LHS of (1) is expressed as a product of n-linear factors.

then the equivalent form of ① will be

$$[p-f_1(x,y)][p-f_2(x,y)]\ldots[p-f_n(x,y)]=0 \quad —②$$

$$\Rightarrow [p-f_1(x,y)]=0,\ [p-f_2(x,y)]=0 \ldots\ldots [p-f_n(x,y)]=0.$$

All these are DE of 1st order & 1st degree. They can be solved by the known methods. If $F_1(x,y,c)=0$, $F_2(x,y,c)=0 \ldots F_n(x,y,c)=0$ respectively represents the sol^n of these eq^{n}s then the general sol^n is given by the product of all these solutions.

· Problems :-

1> solve: $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right)+12=0.$

Sol^n :- $\frac{dy}{dx}=p$

The given eq^n is $p^2-7p+12=0$

i.e $(p-3)(p-4)=0$ or $p=3,4$

we have

$\frac{dy}{dx}=3 \Rightarrow y=3x+c$ or $y-3x-c=0$

Also $\frac{dy}{dx}=4 \Rightarrow y=4x+c$ or $y-4x-c=0$

The general sol^n is given by

$(y-3x-c)(y-4x-c)=0$

2> solve: $y\left(\frac{dy}{dx}\right)^2+(x-y)\cdot\frac{dy}{dx}-x=0$

Sol^n :- The given eq^n is

$yp^2+(x-y)p-x=0.$

$$p=\frac{-(x-y)\pm\sqrt{(x-y)^2+4xy}}{2y}$$

$$p=\frac{(y-x)\pm(x+y)}{2y}$$

i.e $p=\frac{y-\cancel{x}+\cancel{x}+y}{2y}$ $\qquad$ $p=\frac{\cancel{y}-x-x-\cancel{y}}{2y}$

$p=\frac{2y}{2y}$ $\qquad$ $p=\frac{-2x}{2y}$

$p=1$ $\qquad$ $p=-x/y$

we have, $\frac{dy}{dx}=1 \Rightarrow dy=1\cdot dx \Rightarrow y=(x+c)$

or $y-x-c=0$

Also $\frac{dy}{dx}=\frac{-x}{y}$ or $y\cdot dy+x\cdot dx=0$

$\int y\,dy+\int x\,dx=k$

$\frac{y^2}{2}+\frac{x^2}{2}=k.$

$x^2+y^2=2k$

$(x^2+y^2-c)=0$

The general sol^n is given by

$(y-x-c)(x^2+y^2-c)=0$

3> solve: $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$

Sol^n: The given eq^n is

$xyp^2-(x^2+y^2)p+xy=0$

$$p=\frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy}$$

$$p=\frac{(x^2+y^2)\pm(x^2-y^2)}{2xy}$$

i.e $p = \frac{x^2+y^2+x^2-y^2}{2xy}$ or $p = \frac{x^2+y^2-x^2+y^2}{2xy}$

i.e $p = \frac{x}{y}$ or $p = \frac{y}{x}$

we have $\frac{dy}{dx} = \frac{x}{y}$ or $y\,dy - x\,dx = 0$

$\int y\,dy - \int x\,dx = k$

$\frac{y^2}{2} - \frac{x^2}{2} = k$

$y^2 - x^2 = 2k$

$(y^2 - x^2 - c) = 0$

Also $\frac{dy}{dx} = \frac{y}{x}$, or $\frac{dy}{y} - \frac{dx}{x} = 0$

$\int \frac{dy}{y} - \int \frac{dx}{x} = k$

i.e $\log y - \log x = k$ or $\log(y/x) = \log c$

$y = xc \Rightarrow y - cx = 0$

Thus the general sol^n is given by

$(y^2 - x^2 - c)(y - cx) = 0$

4) Solve: $p^2 - 2p\sinh x - 1 = 0$

sol^n: $p^2 - 2p\sinh x - 1 = 0$ by data

$p = \frac{2\sinh x \pm \sqrt{4\sinh^2 x + 4}}{2}$

$p = \frac{2\sinh x \pm 2\sqrt{\cosh^2 x}}{2}$

$p = \sinh x \pm \cosh x$

i.e $p = \sinh x + \cosh x$ or $p = \sinh x - \cosh x$

we have

$\frac{dy}{dx} = \sinh x + \cosh x \Rightarrow y = \cosh x + \sinh x + c$

Also $\frac{dy}{dx} = \sinh x - \cosh x \Rightarrow y = \cosh x - \sinh x + c$

The general sol^n is given by

$(y - \cosh x - \sinh x - c)(y - \cosh x + \sinh x - c) = 0$

5) Solve: $x^2p^2 + xp - (y^2+y) = 0$

sol^n: $x^2p^2 + xp - (y^2+y) = 0$ by data

$p = \frac{-x \pm \sqrt{x^2 + 4x^2(y^2+y)}}{2x^2}$

$= \frac{-x \pm \sqrt{x^2 + 4x^2y^2 + 4x^2y}}{2x^2}$

$= \frac{-x \pm \sqrt{(x+2xy)^2}}{2x^2}$

$p = \frac{-x + x + 2xy}{2x^2}$ or $p = \frac{-x - x - 2xy}{2x^2}$

$p = \frac{-2x - 2xy}{2x^2}$

i.e $p = \frac{y}{x}$ or $p = -\frac{1}{x} - \frac{y}{x}$

we have $\frac{dy}{dx} = \frac{y}{x}$ or $\frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} + k$

i.e $\log y = \log x + \log c$ where $k = \log c$

i.e $\log y = \log(cx) \Rightarrow y - cx = 0$

Also $\frac{dy}{dx} + \frac{y}{x} = \frac{-1}{x}$

This is a linear Eq^n of the form $\frac{dy}{dx} + Py = Q$

whose sol^n is $y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx} \cdot dx + c$

where $P = \frac{1}{x}$ and $Q = \frac{-1}{x}$

Now. $e^{\int P dx} = e^{\int 1/x \, dx} = e^{\log x} = x$

Hence $y \cdot x = \int \frac{-1}{x} \cdot x \, dx + c$ or $xy = -x + c$

or $xy + x - c = 0$

Thus the general sol^n is

$(y - cx)(xy + x - c) = 0$.

Ex solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

sol^n: with the notation

$p = \frac{dy}{dx}$, we have $\frac{1}{p} = \frac{dx}{dy}$.

The given eq^n is

$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ or $\frac{p^2 - 1}{p} = \frac{x}{y} - \frac{y}{x}$

i.e $p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$

$$p = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \sqrt{\left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4}}{2}$$

$$p = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \left(\frac{x}{y} + \frac{y}{x}\right)}{2}$$

$p = \frac{x}{y}$ or $\frac{-y}{x}$

we have, $\frac{dy}{dx} = \frac{x}{y}$ or $x\,dx - y \cdot dy = 0$

$\int x \, dx - \int y \, dy = 0$

$\frac{x^2}{2} - \frac{y^2}{2} = K$

or $x^2 - y^2 - c = 0$ where $c = 2K$

Also $\frac{dy}{dx} = \frac{-y}{x}$ or $\frac{dy}{y} = \frac{-dx}{x}$

$\int \frac{dy}{y} + \int \frac{dx}{x} = K$

i.e $\log y + \log x = K$ or $\log(xy) = \log c$

$xy - c = 0$

where $\log c = K$

Thus the general sol^n is

$(x^2 - y^2 - c)(xy - c) = 0$.

Singular solution :-

Let us suppose that the general solution of the given DE is

$G(x, y, c) = 0$ ——— ①

Treating c as a parameter we differentiate ① partially w.r.to c to obtain.

$\frac{\partial}{\partial c}[G(x, y, c)] = 0$ ——— ②

Eliminating the parameter c from ① & ② we obtain a relation of the form $\phi(x, y) = 0$ and is known as the singular sol^n of the given DE.

Clairaut's Eq^n :-

The eq^n of the form

$y = px + f(p)$ ——— ①

is known as clairaut's eq^n.
The general solution of clairaut's eq^n works out to be

$$y = cx + f(c) \quad \text{——— (2)}$$

Problems :-

1) solve: $y = px + \frac{a}{p}$ and obtain the singular sol^n also.

sol^n: The given eq^n is clairaut's eq^n of the form
$y = px + f(p)$
whose general sol^n is $y = cx + f(c)$
Thus the general sol^n is $y = cx + \frac{a}{c}$

Singular solution :-
Diff partially w.r.to c we have
$0 = x - \frac{a}{c^2}$ or $\frac{a}{c^2} = x$, or $c^2 = \frac{a}{x}$
$c = \sqrt{a/x}$

Hence $y = cx + \left(\frac{a}{c}\right)$ becomes

$$y = \sqrt{\frac{a}{x}} \cdot x + a\sqrt{\frac{x}{a}} \text{ or } y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$$

Thus $y^2 = 4ax$ is the singular solution.

2) obtain the general and singular sol^n of the Eq^n $p = \log(px - y)$

sol^n: $p = \log(px - y)$, by data
or $e^p = px - y$ or $y = px - e^p$
Since $y = px - e^p$ is in the form of clairaut's Eq^n $y = px + f(p)$

The general sol^n is $y = cx + f(c)$
Thus the general sol^n is $y = cx - e^c$
Now diff w.r.to c partially we have
$0 = x - e^c$ or $e^c = x$
$\Rightarrow c = \log x$
Thus the singular sol^n is $y = x\log x - x$

3) S.T the Eq^n $xp^2 + px - py + 1 - y = 0$ is clairaut's Eq^n hence obtain the general and singular sol^n

$\Rightarrow xp^2 + px - py + 1 - y = 0$ by data
i.e $xp^2 + px + 1 = y(p+1)$
i.e $y = \frac{xp(p+1) + 1}{p+1}$

or $y = px + \frac{1}{p+1}$ ——— (1)

(1) is in the clairaut's form $y = px + f(p)$
whose general sol^n is $y = cx + f(c)$
Thus the general sol^n is $y = cx + \frac{1}{c+1}$

Now Diff partially w.r.to c we have
$0 = x - \frac{1}{(c+1)^2}$ or $(c+1)^2 = \frac{1}{x} \Rightarrow (c+1) = \frac{1}{\sqrt{x}}$ or $c = \frac{1}{\sqrt{x}} - 1$

Hence the general sol^n becomes
$$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \sqrt{x}$$

i.e $y = \sqrt{x} - x + \sqrt{x}$ or $x + y = 2\sqrt{x} \Rightarrow (x+y)^2 = 4x$

Thus the singular sol^n is $(x+y)^2 = 4x$

4> Solve $(y-px)^2 = 4p^2+9$

Sol^n: we have $(y-px)^2 = 4p^2+9$

$\Rightarrow y - px = \sqrt{4p^2+9}$

or $y = px + \sqrt{4p^2+9}$

The eq^n is being in the Clairaut's form $y = px + f(p)$ has the general sol^n $y = cx + f(c)$

The required general sol^n is $y = cx + \sqrt{4c^2+9}$

5> Obtain the general sol^n & singular sol^n of the following eq^n as Clairaut's eq^n $xp^3 - yp^2 + 1 = 0$

Sol^n: $xp^3 - yp^2 + 1 = 0$ by data

i.e $yp^2 = xp^3 + 1$ or $y = \frac{xp^3+1}{p^2}$

i.e $y = px + \frac{1}{p^2}$ is in the Clairaut's form $y = px + f(p)$

whose general sol^n is $y = cx + \frac{1}{c^2}$

Now diff partially w.r.to c we have

$0 = x - \frac{2}{c^3}$ or $c^3 = \frac{2}{x} \Rightarrow c = \left(\frac{2}{x}\right)^{1/3}$

Hence the general sol^n becomes

$y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3}$ or $y = 2^{1/3} \cdot x^{2/3} + \frac{x^{2/3}}{2^{2/3}}$

i.e $2^{2/3} \cdot y = 2 \cdot x^{2/3} + x^{2/3}$

or

$2^{2/3} \cdot y = 3x^{2/3}$

Equivalently $(2^{2/3} y)^3 = (3x^{2/3})^3$ or $4y^3 = 27x^2$

Thus the singular sol^n is $4y^3 = 27x^2$

Equations reducible to the Clairaut's form.

Ex. Solve the eq^n $(px-y)(py+x) = 2p$ by reducing into Clairaut's form taking the substitution $X = x^2$, $Y = y^2$

$\Rightarrow X = x^2 \Rightarrow \frac{dX}{dx} = 2x$

$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y$

Now, $p = \frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx}$ and let $P = \frac{dY}{dX}$

i.e $p = \frac{1}{2y} \cdot P \cdot 2x$ or $p = \frac{x}{y} \cdot P$

i.e $p = \frac{\sqrt{X}}{\sqrt{Y}} \cdot P$

Consider, $(px-y)(py+x) = 2p$

i.e $\left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X} - \sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}} \cdot P\sqrt{Y} + \sqrt{X}\right] = \frac{2\sqrt{X}}{\sqrt{Y}} \cdot P$

or $\frac{(PX-Y)}{\sqrt{Y}} \cdot (P+1)\sqrt{X} = \frac{2\sqrt{X}}{\sqrt{Y}} \cdot P$

i.e $(PX-Y)(P+1) = 2P$ or $PX - Y = \frac{2P}{P+1}$

i.e $Y = PX - \frac{2P}{P+1}$

This is in the Clairaut's form and hence the associated general solution is

$Y = cX - \frac{2c}{c+1}$ or $y^2 = cx^2 - \frac{2c}{c+1}$

Thus the required general solution of the given eq^n is $y^2 = cx^2 - \frac{2c}{c+1}$

Q. 2) Solve: $e^{4x}(p-1)+e^{2y}.p^2=0$ by using the substitution $u=e^{2x}$ and $v=e^{2y}$

Sol^n :→ $u=e^{2x} \Rightarrow \frac{du}{dx} = 2\cdot e^{2x}$

$$v = e^{2y} \Rightarrow \frac{dv}{dy} = 2\cdot e^{2y}$$

Now, $p = \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}$ and let $P = \frac{dv}{du}$

i.e, $p = \frac{1}{2\cdot e^{2y}} P\cdot 2\cdot e^{2x}$ or $p = \frac{u}{v}P$

Consider, $e^{4x}(p-1)+e^{2y}.p^2=0$

i.e $u^2\left(\frac{u}{v}P-1\right)+v.\frac{u^2}{v^2}P^2=0$

i.e $u^2\frac{(uP-v)}{v}+\frac{u^2P^2}{v}=0$

or $uP-v+P^2=0$

i.e $v = Pu+P^2$. is in the ~~Cla~~ Clairaut' form and hence the associated general sol^n is

$v = cu+c^2$ or $e^{2y}=c.e^{2x}+c^2$

Thus the required general solution of the given Equation is $e^{2y}=c\cdot e^{2x}+c^2$

3) Find the general and singular of the equation $x^2(y-px)=p^2y$ by reducing into clairaut form using the substitution $X=x^2$, $Y=y^2$

Sol^n :⇒ This problem is similar to 1st proceeding on the same lines we have, $p=\frac{\sqrt{X}}{\sqrt{Y}}\cdot P$ where $P=\frac{dY}{dX}$

The given eq^n $x^2(y-px)=p^2y$ becomes.

$$X\left(\sqrt{Y}-\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}\right)=\frac{X}{Y}P^2\sqrt{Y}$$

i.e $X\frac{(Y-PX)}{\sqrt{Y}} = \frac{XP^2}{\sqrt{Y}}$ or $Y-PX=P^2$

i.e $Y=PX+P^2$ is in the clairaut's form

The associated generated sol^n is $Y=cX+c^2$

Thus the required general sol^n of the given Eq^n is $y^2=cx^2+c^2$

Now diff w.r.to c partially we have,

$0=x^2+2c$ or $c=-x^2/2$

Hence the general sol^n becomes

$y^2=-\frac{x^4}{2}+\frac{x^4}{4}$ or $4y^2=-x^4$

Thus the required singular sol^n is $x^4+4y^2=0$