Taylor's and Maclaurin's Series expansion for a function of one variable.

The infinite series expansion of a differentiable fun^n $y = f(x)$ about the point, $x = a$ will be a Series in powers of $(x-a)$ known as Taylor's Series expansion. In particular if $a = 0$, the Series will be in ascending power of x, known as Maclaurin's Series expansion

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ---$$

This is called Taylor's series expansion of $f(x)$ about the point 'a'.

In particular, if $a = 0$ we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ---$$

This is called Maclaurins Series expansion of $f(x)$.

Note :-

For convenience we shall use the following alternative notation.

$y(x)$ for $f(x)$ and $y_1(x)$, $y_2(x)$, $y_3(x)$ ----- respectively for $f'(x)$, $f''(x)$, $f'''(x)$ --- so that we have,

$$y(x) = y(a) + (x-a) y_1(a) + \frac{(x-a)^2}{2!} y_2(a) + ----$$

(Taylor's expansion)

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \cdots$$

(Maclaurin's expansion)

We need to find successive derivatives of the given $y(x)$ and evaluate them at the given point $x=a$ for obtaining the Taylor's expansion and evaluate at $x=0$ for obtaining the Maclaurin's expansion.

Problems.

1) Obtain the Taylor's expansion of $\log_e x$ about $x=1$ upto the term containing 4th degree and hence obtain $\log_e(1.1)$.

Sol.^{n} :-> we have Taylor's expansion about $x=a$ given by

$$y(x) = y(a) + (x-a) y_1(a) + \frac{(x-a)^2}{2!} y_2(a) + \cdots$$

By data, $y(x) = \log_e x$; $a=1$. $x=a$ $(x-a)_{x=a}$ $x=0$ (x)

$y(1) = \log_e 1 = 0$

Diff $y(x)$ successively we get

$y_1(x) = \frac{1}{x}$ $\therefore y_1(1) = 1$, $y_2(x) = -\frac{1}{x^2}$, $y_2(1) = -1$

$y_3(x) = \frac{2}{x^3}$ $\therefore y_3(1) = 2$, $y_4(x) = -\frac{6}{x^4}$ $\therefore y_4(1) = -6$

Taylor's series upto fourth degree term with $a=1$ is given by

$$y(x) = y(1) + (x-1) y_1(1) + \frac{(x-1)^2}{2!} y_2(1) + \frac{(x-1)^3}{3!} y_3(1) + \frac{(x-1)^4}{4!} y_4(1)$$

i.e $\log_e x = 0 + (x-1)\cdot 1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{\cancel{6}\,3}(\cancel{2}) + \frac{(x-1)^4 \cdot (-\cancel{6})}{\cancel{24}\,4}$

Thus, $\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$

we shall now substitute $x=1.1$ to obtain $\log(1.1)$

Also,

$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.0953$

Thus, $\log_e(1.1) = 0.0953$

2) Expand $\tan^{-1}x$ in powers of $(x-1)$ upto the term containing fourth degree.

Sol.^{n} :-> Taylor's expansion in powers of $(x-1)$ is given by

$$y(x) = y(1) + (x-1) y_1(1) + \frac{(x-1)^2}{2!} y_2(1) + \frac{(x-1)^3}{3!} y_3(1) + \frac{(x-1)^4}{4!} y_4(1) + \cdots$$

$y = \tan^{-1}x$ $\therefore y(1) = \tan^{-1}1 = \frac{\pi}{4}$

$y_1 = \frac{1}{(1+x^2)}$ $\therefore y_1(1) = \frac{1}{2}$

we have $(1+x^2) y_1 = 1$

$\log 2^{-1} = -1\log 2 = -\log 2$

we have to successively differentiate to obtain expressions involving y_2, y_3, y_4 and evaluate them at $x=1$.

Hence we have on differentiating (1)

$$(1+x^2)y_2 + 2xy_1 = 0 \quad \text{——— (2)}$$

putting $x=1$; $2y_2(1) + 2\cdot 1\cdot \frac{1}{2} = 0$

$$\therefore y_2(1) = -1/2$$

Diff (2) w.r to x we get

$$(1+x^2)y_3 + 4xy_2 + 2y_1 = 0 \quad \text{——— (3)}$$

putting $x=1$, $2y_3(1) - 2 + 1 = 0$ $\therefore y_3(1) = 1/2$

Diff (3) w.r. to x we get

$$(1+x^2)y_4 + 6xy_3 + 6y_2 = 0 \quad \text{——— (4)}$$

putting $x=1$, $2y_4(1) + 3 - 3 = 0$ $\therefore y_4(1) = 0$

we shall substitute these values in the Expansion of $y(x)$

Thus,

$$\tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}\left\{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6}\right\}$$

3) obtain Taylor's series expansion of $\log(\cos x)$ about the point $x = \pi/3$ upto the fourth degree of term.

Sol^n:⇒ Taylor's expansion of $y(x)$ about $x = \pi/3$ is given by

$$y(x) = y(\pi/3) + (x-\pi/3)\,y_1(\pi/3) + \frac{(x-\pi/3)^2}{2!}\,y_2(\pi/3) + \frac{(x-\pi/3)^3}{3!}\,y_3(\pi/3) + \frac{(x-\pi/3)^4}{4!}\,y_4(\pi/3) + \ldots \quad \text{——— (1)}$$

Let $y(x) = \log(\cos x)$

$\therefore y(\pi/3) = \log(1/2) = -\log 2$

$y_1 = \frac{1}{\cos x}\cdot -\sin x$

i.e $y_1 = -\tan x$. $\therefore y_1(\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$y_2 = -\sec^2 x = -(1+\tan^2 x)$

i.e $y_2 = -(1+y_1^2)$ $\therefore y_2(\pi/3) = -(1+(\sqrt{3})^2) = -4$

$y_3 = -2y_1y_2$ $\therefore y_3(\pi/3) = -2\cdot -\sqrt{3}\cdot -4 = -8\sqrt{3}$

$y_4 = -2[y_1y_3 + y_2^2]$ $\therefore y_4(\pi/3) = -2[-\sqrt{3}\cdot -8\sqrt{3} + 16] = -80$

substituting these values in (1) we have

$$\log(\cos x) = -\log 2 - (x-\pi/3)\sqrt{3} - \frac{(x-\pi/3)^2}{2}\cdot 4 - \frac{(x-\pi/3)^3}{6}\cdot 8\sqrt{3} - \frac{(x-\pi/3)^4}{24}\cdot 80 + \ldots$$

Thus

$$\log(\cos x) = -\log 2 - \sqrt{3}(x-\pi/3) - 2(x-\pi/3)^2 - \frac{4}{\sqrt{3}}(x-\pi/3)^3 - \frac{10}{3}(x-\pi/3)^4$$

4) obtain the Maclaurin's Expansion of $\sin^{-1}x$ upto the term containing x^5.

Sol^n:⇒ we have $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \ldots$

$y = \sin^{-1}x$

$y(0) = \sin^{-1}(0) = 0$

$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow y_1(0) = \frac{1}{\sqrt{1-0}} = \frac{1}{\sqrt{1}} = 1$

$y_1^2 = \frac{1}{1-x^2} \Rightarrow y_1^2(1-x^2) = 1$

Diff w.r to x

$(1-x^2)\cdot 2y_1\cdot y_2 + y_1^2(-2x) = 0$ (÷ by $2y_1$)

we get $(1-x^2)y_2 - xy_1 = 0$ —— (2)

Now putting $x=0$, we have

$1 \cdot y_2(0) - 0 = 0$, $y_2(0) = 0$

Now diff (2) w.r.to x we obtain,

$(1-x^2)y_3 + y_2(-2x) - [xy_2 + y_1 \cdot 1] = 0$

i.e $(1-x^2)y_3 - 3xy_2 - y_1 = 0$ —— (3)

putting $x=0$, we have $y_3(0) - 0 - 1 = 0$

$y_3(0) = 1$

Diff (3) again w.r.to x we obtain

$(1-x^2)y_4 + y_3(-2x) - 3(xy_3 + y_2 \cdot 1) - y_2 = 0$

i.e $(1-x^2)y_4 - 5xy_3 - 4y_2 = 0$ —— (4)

putting $x=0$, we have, $y_4(0) - 0 - 0 = 0$

$y_4(0) = 0$

Diff (4) again w.r.to x we obtain

$(1-x^2)y_5 + y_4(-2x) - 5(xy_4 + y_3 \cdot 1) - 4y_3 = 0$

i.e $(1-x^2)y_5 - 7xy_4 - 9y_3 = 0$ —— (5)

putting $x=0$, we have, $y_5(0) - 0 - 9 \cdot 1 = 0$

$y_5(0) = 9$

Substituting these values in the expansion of $y(x)$ we have,

$\sin^{-1}x = 0 + x \cdot 1 + \frac{x^2}{2} \cdot 0 + \frac{x^3}{6} \cdot 1 + \frac{x^4}{24} \cdot 0 + \frac{x^5}{120} \cdot 9$

Thus. $\boxed{\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40}}$

(5) Expand $e^{\sin x}$ as Maclaurin's Series upto the terms containing x^4

sol^n:- we have Maclaurin's expansion

$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$

Let $y = e^{\sin x}$ $\Rightarrow$ $y(0) = e^0 = 1$

$y_1 = e^{\sin x} \cdot \cos x$ or $y_1 = y \cdot \cos x$ $\Rightarrow y_1(0) = 1 \cdot 1 = 1$

$y_2 = -y\sin x + \cos x \cdot y_1$ $\Rightarrow y_2(0) = 0 + 1 = 1$

$y_3 = -(y\cos x + y_1 \sin x) + (\cos x \cdot y_2 - y_1 \sin x)$

$= -y_1 - 2y_1 \sin x + \cos x \cdot y_2$

$\therefore y_3(0) = -1 - 0 + 1 = 0$

$y_4 ~~(0)~~ = -y_2 - 2(y_1\cos x + \sin x\, y_2) + (\cos x\, y_3 - \sin x\, y_2)$

$= -y_2 - 2y_1\cos x - 3\sin x\, y_2 + \cos x \cdot y_3$

$y_4(0) = -1 - 2 - 0 + 0 = -3$

$\therefore y_4(0) = -3$

Thus by substituting these values in the Expansion of $y(x)$ we get

$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$

(6) Expand $\log(1+\sin x)$ in powers of x upto the term x^4

sol^n:- we have Maclaurin's Expansion in powers of x,

$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$

Consider, $y = \log(1+\sin x)$ $\Rightarrow y(0) = \log 1 = 0$

$y_1 = \frac{\cos x}{1+\sin x}$ $\Rightarrow \therefore y_1(0) = 1$

i.e $(1+\sin x)y_1 = \cos x$ —— (1)

Diff w.r.to x we get
$(1+\sin x)y_2 + \cos x\, y_1 = -\sin x$ —— ②
At $x=0$, we get $y_2(0)+1=0$
$\therefore y_2(0) = -1$

Diff ② again w.r.to x we get
$(1+\sin x)y_3 + 2y_2\cos x - y_1\sin x = -\cos x$ —— ③
At $x=0$, we get, $y_3(0) - 2 - 0 = -1$
$\therefore y_3(0) = 1$

Diff ③ again we get
$(1+\sin x)y_4 + \cos x\, y_3 + 2(-y_2\sin x + \cos x\, y_3)$
$-(y_1\cos x + \sin x\, y_2) = \sin x$
ie $(1+\sin x)y_4 + 3\cos x\cdot y_3 - 3y_2\sin x - y_1\cos x = \sin x$
At $x=0$, we get $y_4(0) + 3 - 0 - 1 = 0$.
$y_4(0) = -2$

Thus by substituting these values in the Expansion we get
$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

⑦ Expand $\log(\sec x)$ upto the term containing x^6 using Maclaurin's series. or
Expand $\log(\sec x)$ in ascending powers of x upto the first three non vanishing terms.

Sol^n :- $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots$
$y = \log(\sec x)$
$y(0) = \log 1 = 0$
$y_1 = \frac{\sec x \tan x}{\sec x}$ $\quad y_1 = \tan x \quad y_1(0) = 0$

$y_2 = \sec^2 x \qquad y_2(0) = 1$
Now $y_2 = 1 + \tan^2 x = 1 + y_1^2$
Diff w.r to x Successively we have
$y_3 = 2y_1 y_2 \qquad y_3(0) = 0\cdot 1 = 0$
$y_4 = 2[y_1 y_3 + y_2\cdot y_2]$
$= 2[y_1 y_3 + y_2^2] \qquad y_4(0) = 2[0\cdot 0 + 1] = 2$
$y_5 = 2[y_1\cdot y_4 + y_3\cdot y_2 + 2y_2\cdot y_3]$
$= 2[y_1\cdot y_4 + 3y_2\cdot y_3] \qquad y_5(0) = 2[0+0] = 0$
$y_6 = 2[y_1\cdot y_5 + y_4\cdot y_2 + 3(y_2\cdot y_4 + y_3\cdot y_3)]$
$\left\{ y_6 = 2[y_1 y_5 + 4\cdot y_2 y_4 + y_3^2] \right.$
$\left. y_6 = 2[0 + 4(1)(1) + 0] \right\}$
$y_6 = 2y_1 y_5 + 2y_4\cdot y_2 + 6y_2 y_4 + 6y_3^2$
$\cancel{y_6} = \cancel{0+2}$
$y_6 = 2y_1 y_5 + 8y_2 y_4 + 6y_3^2$
$= 2(0) + 8(2) + 6\,0$
$y_6 = 16$
Substituting these values in the Expansion of $y(x)$ we get
$$\log(\sec x) = \frac{x^2}{2}\cdot 1 + \frac{x^4}{24}\cdot 2 + \frac{x^6}{720}\,16$$
Thus $\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$

⑧ Expand $\tan^{-1}(1+x)$ as far as the term containing x^3

Sol^n :→ $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots$
Consider, $y = \tan^{-1}(1+x) \qquad \therefore y(0) = \tan^{-1}(1) = \pi/4$
$y_1 = \frac{1}{1+(1+x)^2} = \frac{1}{x^2+2x+2} \qquad y_1(0) = \frac{1}{2}$

i.e $(x^2+2x+2)y_1=1$ —— ①

Diff w.r.to x

$(x^2+2x+2)y_2+2(x+1)y_1=0$ —— ②

At $x=0$ we get $2y_2(0)+2(1/2)=0$ $\quad y_2(0)=-1/2$

Diff ② w.r.to x we have

$(x^2+2x+2)y_3+2(x+1)y_2+2(x+1)y_2+2y_1=0$

i.e $(x^2+2x+2)y_3+4(x+1)y_2+2y_1=0$ —— ③

At $x=0$, we get $2y_3(0)+4(-1/2)+2(1/2)=0$

$\therefore y_3(0)=1/2$

Thus by substituting these values in the Expansion of $y(x)$ we get

$$\tan^{-1}(1+x)=\frac{\pi}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{12}$$

⑨ Using Maclaurin's series P.T $\sqrt{1+\sin 2x}=1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}$

Sol^n:- we have $y(x)=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\ldots$

Let $y=\sqrt{1+\sin 2x}=\sqrt{\cos^2 x+\sin^2 x+2\sin x\cdot\cos x}$

$=\sqrt{(\cos x+\sin x)^2}=\cos x+\sin x$

Hence $y=\cos x+\sin x$ $\quad y(0)=1$

$y_1=-\sin x+\cos x$ $\quad y_1(0)=1$

$y_2=-\cos x-\sin x=-y$

i.e $y_2=-y$ $\quad y_2(0)=-1$

$y_3=-y_1$ $\quad y_3(0)=-1$

$y_4=-y_2$ $\quad y_4(0)=1$

Thus by substituting these values in the Expansion of $y(x)$ we get

$$\sqrt{1+\sin 2x}=1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\ldots$$

10) obtain the Maclaurin's expansion of the function $\log(1+x)$ and hence deduce that

$$\log\sqrt{\frac{1+x}{1-x}}=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$$

Sol^n:⇒ we have $y(x)=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)$

Let $y=\log(1+x)$ $\quad \therefore y(0)=\log 1=0$

$y_1=\frac{1}{1+x}$ $\quad y_1(0)=1$

$y_2=\frac{-1}{(1+x)^2}$ $\quad y_2(0)=-1$

$y_3=\frac{2}{(1+x)^3}$ $\quad y_3(0)=2$

$y_4=\frac{-6}{(1+x)^4}$ $\quad y_4(0)=-6$

$y_5=\frac{24}{(1+x)^5}$ $\quad y_5(0)=24$

by substituting these values we get

$$\log(1+x)=0+x\cdot 1+\frac{x^2}{2}(-1)+\frac{x^3}{6}\cdot 2+\frac{x^4}{24}(-6)+\frac{x^5}{120}\cdot 24-$$

Thus $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots$ —— ①

Next $\log\sqrt{\frac{1+x}{1-x}}=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=\frac{1}{2}(\log(1+x)-\log(1-x))$ —— ②

changing x to $-x$ in ① we obtain

$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots$ —— ③

using (1) and (3) in (2) we have

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left\{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right) - \left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right)\right\}$$

i.e $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left(2x+2\frac{x^3}{3}+2\frac{x^5}{5}\cdots\right) = x+\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

Thus $\log\sqrt{\frac{1+x}{1-x}} = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$

11> Expand $\tan\left(\frac{\pi}{4}+x\right)$ upto the 4th degree terms

Sol^n: $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$

Let $y = \tan(\pi/4 + x)$ $\qquad y(0) = \tan\pi/4 = 1$

$y_1 = \sec^2(\pi/4+x) = 1+y^2$ $\qquad y_1(0) = 2$

$y_2 = 2yy_1$ $\qquad y_2(0) = 4$

$y_3 = 2(yy_2 + y_1^2)$ $\qquad y_3(0) = 2(4+4) = 16$

$y_4 = 2(yy_3 + 3y_1 \cdot y_2)$ $\qquad y_4(0) = 2(16+24) = 80$

Substituting these values in the expansion of $y(x)$ we have

$$\tan(\pi/4+x) = 1 + x\cdot 2 + \frac{x^2}{2}\cdot 4 + \frac{x^3}{6}\cdot 16 + \frac{x^4}{24}\cdot 80$$

Thus $\tan(\pi/4+x) = 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4$

12> Expand $e^{x\sin x}$ in ascending powers of x upto the fourth degree terms

Sol^n: $y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots$

Let $y = e^{x\sin x}$ $\qquad \therefore y(0) = 1$

$y_1 = e^{x\sin x}(x\cos x + \sin x)$ $\qquad y_1(0) = 0$

i.e $y_1 = y(x\cos x + \sin x)$

Now, $y_2 = y(-x\cdot\sin x + 2\cos x) + y_1(x\cos x + \sin x)$

Hence, $y_2(0) = 1(0+2)+0$ $\qquad y_2(0) = 2$

Next, $y_3 = y(-x\cdot\cos x - 3\sin x) + y_1(-x\sin x + 2\cos x) + y_1(-x\sin x + 2\cos x) + y_2(x\cos x + \sin x)$

i.e $y_3 = y(-x\cos x - 3\sin x) + 2y_1(-x\sin x + 2\cos x) + y_2(x\cos x + \sin x)$

Hence, $y_3(0) = 0+0+0$ $\qquad \therefore y_3(0) = 0$

Next, $y_4 = y(x\sin x - 4\cos x) + y_1(-x\cos x - 3\sin x) + 2y_1(-x\cos x - 3\sin x) + 2y_2(-x\sin x + 2\cos x) + y_2(-x\sin x + 2\cos x) + y_3(x\cos x + \sin x)$

Hence, $y_4(0) = -4+0+0+8+4 = 8$ $\qquad \therefore y_4(0) = 8$

Thus by substituting these values in the expansion of $y(x)$ we get

$$e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \cdots$$

13> Obtain the Maclaurin's Expansion of $\log(1+e^x)$ as far as the fourth degree terms

Sol^n: $y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots$

Let $y = \log(1+e^x)$

$y(0) = \log 2$

$y_1 = \frac{e^x}{1+e^x}$ $\qquad y_1(0) = \frac{1}{2}$

i.e $(1+e^x)y_1 = e^x$ ——— (1)

Diff w.r to x we get

$(1+e^x)y_2 + e^x y_1 = e^x$ —— (2)

At $x=0$, $2y_2(0) + \frac{1}{2} = 1$ $\therefore y_2(0) = \frac{1}{4}$

Diff (2) w.r to x we get

$(1+e^x)y_3 + 2e^x y_2 + e^x y_1 = e^x$ —— (3)

At $x=0$, $2y_3(0) + 1/2 + 1/2 = 1$ $y_3(0) = 0$

Diff (3) w.r to x we get

$(1+e^x)y_4 + 3e^x y_3 + 3e^x y_2 + e^x y_1 = e^x$ —— (4)

At $x=0$, $2y_4(0) + \frac{3}{4} + \frac{1}{2} = 1$ $y_4(0) = -\frac{1}{8}$

Thus by substituting these values in the expansion of $y(x)$ we get

$$\log(1+e^x) = \log_e 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}$$

Similar problem :⇒

Expand $\frac{e^x}{1+e^x}$ using Maclaurin's series upto and including third degree terms.

Sol^n :⇒ The given $y = \frac{e^x}{1+e^x}$ is same as 'y_1' of the earlier problem $y(0) = 1/2$ and proceeding on the same lines the other values will be

$y_1(0) = 1/4$, $y_2(0) = 0$, $y_3(0) = -1/8$

Thus the required expansion is given by

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48}$$

(14) Expand $\log(1+\cos x)$ by Maclaurin's series upto the term containing x^4

Sol^n :⇒ $y(x) = \log(1+\cos x)$, $y(0) = \log 2$

we prefer to simplify the given functions

$y = \log(1+\cos x) = \log[2\cos^2(x/2)]$

i.e $y = \log 2 + 2\log\cos(x/2)$ $\therefore y(0) = \log 2$

Now, $y_1 = -\tan(x/2)$ $y_1(0) = 0$

$y_2 = -(1/2)\sec^2(x/2)$ $y_2(0) = -1/2$

Also, $y_2 = -\frac{1}{2}[1+\tan^2(x/2)] = -\frac{1}{2}(1+y_1^2)$

$y_3 = -\frac{1}{2}(2y_1 y_2) = -y_1 y_2$ $\therefore y_3(0) = 0$

$y_4 = -y_1 y_3 - y_2^2$ $y_4(0) = -1/4$

Thus, $\log_e(1+\cos x) = \log_e 2 - \frac{x^2}{4} - \frac{x^4}{96}$

15> Obtain the Maclaurin's expansion of a^x

Sol^n :- $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \dots$

Let $y = a^x$ $\therefore y(0) = 1$

$y_1 = a^x \log a = y\log a$ $\therefore y_1(0) = \log a$

$y_2 = y_1 \log a$ $\therefore y_2(0) = (\log a)^2$

$y_3 = y_2 \log a$ $\therefore y_3(0) = (\log a)^3$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \dots$$

16> Expand $\tan^{-1}x$ in ascending powers of x upto the first three non-zero terms and hence S.T $\pi = 4(1 - 1/3 + 1/5 - \dots)$

Sol^n: let $y = \tan^{-1}x$ $\quad\therefore y(0) = \tan^{-1}(0) = 0$

$y_1 = \frac{1}{1+x^2}$ $\quad\therefore y_1(0) = 1$

i.e $(1+x^2)y_1 = 1$ — (1)

Diff w.r to x we get

$(1+x^2)y_2 + 2xy_1 = 0$ — (2)

at $x=0$, we get, $y_2(0) = 0$.

Diff (2) again w.r to x we get

$(1+x^2)y_3 + 2xy_2 + 2xy_2 + 2y_1 = 0$

i.e $(1+x^2)y_3 + 4xy_2 + 2y_1 = 0$ — (3).

At $x=0$, we get $y_3(0) = -2$, $y_3(0) = -2$

Diff (3) w.r to x we get

$(1+x^2)y_4 + 2xy_3 + 4xy_3 + 4y_2 + 2y_2 = 0$

i.e $(1+x^2)y_4 + 6xy_3 + 6y_2 = 0$ — (4)

At $x=0$, we get $y_4(0) = 0$.

Diff (4) w.r to x

$(1+x^2)y_5 + 2xy_4 + 6xy_4 + 6y_3 + 6y_3 = 0$

i.e $(1+x^2)y_5 + 8xy_4 + 12y_3 = 0$ — (5)

At $x=0$, $y_5(0) = 24$

$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

putting $x=1$, we get $\tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

i.e $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5}$ Thus, $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

Note :⇒

Maclaurin's expansions of the functions $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, e^x can be found easily and it is advisible to remember them. They are as follows.

i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

iii) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

iv) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

v) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Assignment :⇒

1) Expand $\tan x$ about the point $x = \pi/4$ upto the third degree terms and hence find $\tan 46°$

2) show that

$$\sqrt{x} = \sqrt{2}\left[1 + \frac{(x-2)}{4} - \frac{(x-2)^2}{32} + \frac{(x-2)^3}{128} - \cdots\right].$$

* Obtain the Maclaurin's expansion of the following functions as indicated.

3) $e^{\cos x}$ upto the fourth degree terms

4) $\log(\sec x + \tan x)$ upto the first 3 non vanishing terms

5) $\log(1 + \tan x)$ upto the 3rd degree terms

Indeterminate forms :→

If an expression $F(x)$ at $x=a$ assumes forms like $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$ which donot represent any value are called Indeterminate forms.

The concept of limit gives a meaningful value for the function $F(x)$ at $x=a$. overcoming these indeterminate forms.

→ The reader is familiar with the evaluation of limit in the cases of $0/0$ or ∞/∞ without the involvement of differentiation.

→ Few more indeterminate forms : $\infty-\infty$, $\infty\times 0$, ∞^0; 1^∞, 0^0 can be reduced to the two basic indeterminate forms. $0/0$ and ∞/∞. Then limit is found passing through a process of differentiation warranted by an established simple rule called L' Hospital's rule.

L' Hospital's rule (Theorem) :→

Statement :→

If $f(x)$ and $g(x)$ are two functions such that

i) $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ or $f(a)=0=g(a)$

ii) $f'(x)$ and $g'(x)$ exist and $g'(a)\neq 0$, then.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note :- Extension of the theorem.

If $f'(a)=0$ and $g'(a)=0$ then we have

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on.

→ Note for solving problems by L' Hospital's rule :-

The rule is applicable for the form $0/0$. It can also be applied for the form ∞/∞ as we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left\{\frac{1/g(x)}{1/f(x)}\right\}$$

where $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$.

However while applying the rule in this case also, we follow the usual procedure of differentiating the numerator $f(x)$ and the denominator $g(x)$ separately. If the indeterminate form persists after applying the rule once, we can apply the rule repeatedly till we arrive at a definite value. It is highly advisible to look for simplification at each stage.

Remark : problems have been bifurcated into four types and the procedure too has been explained seperately in each type. The following four standard limits and well known simple properties connected with limits can be readily used.

v) $\lim_{x\to 0} \frac{a^x-1}{x} = \log a$ vi) $\lim_{x\to 0} \frac{e^x-1}{x} = \log e = 1$ vii) $\lim_{x\to 0} (1+x)^{1/x} = e$

i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ii) $\lim_{x\to 0} \frac{x}{\sin x} = 1$

iii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$ iv) $\lim_{x\to 0} \frac{x}{\tan x} = 1$

viii) $\lim_{x\to 0} \frac{\log(1+x)}{x} = \log e = 1$

Type - 1

The rule can be applied directly in the case of forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. In the cases of $\infty-\infty$ and $\infty \times 0$, we have to employ simple methods to simplify the given expression in bringing it to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that the L' Hospital's rule can be employed.

Problems :-

Evaluate the following limit.

1) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

2) $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$

3) $\lim_{x\to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$

4) $\lim_{x\to 0} \frac{a^x - b^x}{x}$

5) $\lim_{x\to 0} \frac{\sinh x - x}{\sin x - x\cos x}$

6) $\lim_{x\to a} \frac{x^a - a^x}{x^a - a^a}$

7) $\lim_{x\to \pi/2} \frac{\log(x-\pi/2)}{\tan x}$

8) $\lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x}$.

$\log x = \frac{1}{x}$ $\frac{d}{dx}\left(\frac{u}{v}\right) =$

$\frac{d}{dx}(uv) = uv' + vu'$ $xe^x = xe^x + e^x$

1) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

Sol^n: let $K = \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule.

$K = \lim_{x\to 0} \frac{x\cdot e^x + e^x - 1/1+x}{2x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{xe^x + e^x + e^x + 1/(1+x)^2}{2}$

$= \frac{0+1+1+1}{2} = \frac{3}{2}$

$1/x$

$-\frac{1}{x^2}$

$\frac{1}{(1+0)} = \frac{1}{1}$

Thus $K = 3/2$.

2) $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$

Let $K = \lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule.

$= \lim_{x\to 0} \frac{\sin x}{x\cdot(1/1+x) + \log(1+x)}$ --- $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\cos x}{x\cdot -1/(1+x)^2 + (1/1+x) + (1/1+x)}$

$= \frac{1}{0+1+1} = \frac{1}{2}$ $K = \frac{1}{2}$

3) Let $K = \lim_{x\to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule.

$K = \lim_{x\to \pi/2} \frac{\cos x/\sin x}{-2(\pi/2 - x)}$

i.e $= \lim_{x \to \pi/2} \frac{\cot x}{-2(\pi/2 - x)} \; \cdots \left(\frac{0}{0}\right)$

Now $K = \lim_{x \to \pi/2} \frac{-\operatorname{cosec}^2 x}{2} = \frac{-1}{2}$

Thus $K = -1/2$

4) $\lim_{x \to 0} \frac{a^x - b^x}{x}$

Let $K = \lim_{x \to 0} \frac{a^x - b^x}{x} \; \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$K = \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1}$

$= \log a - \log b = \log(a/b)$

Thus $K = \log(a/b)$

5) $\lim_{x \to 0} \frac{\sinh x - x}{\sin x - x\cos x} \; \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$K = \lim_{x \to 0} \frac{\cosh x - 1}{\cos x + x\sin x - \cos x}$

$= \lim_{x \to 0} \frac{\cosh x - 1}{x \sin x} \; \cdots \left(\frac{0}{0}\right)$

$\therefore K = \lim_{x \to 0} \frac{\sinh x}{x\cos x + \sin x} \; \cdots \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\cosh x}{-x\sin x + \cos x + \cos x} = \frac{1}{2}$

Thus $K = \frac{1}{2}$

(applying the rule thrice)

6) $\lim_{x \to a} \frac{x^x - a^x}{x^a - a^a} \; \cdots \; f$

Solⁿ: Let $K = \lim_{x \to a} \frac{x^x - a^x}{x^a - a^a} \; \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$K = \lim_{x \to a} \frac{x^x(1 + \log x) - a^x \log a}{a x^{a-1} - 0}$

$K = \frac{a^a(1 + \log a) - a^a \log a}{a \cdot a^{a-1}} = \frac{a^a}{a^a} = 1$

Thus $K = 1$

7) Let $K = \lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x} \; \cdots \left(\frac{-\infty}{\infty}\right)$

Applying L' Hospital's rule

$K = \lim_{x \to \pi/2} \frac{1/(x - \pi/2)}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\cos^2 x}{(x - \pi/2)} \; \cdots \left(\frac{0}{0}\right)$

$\therefore K = \lim_{x \to \pi/2} \frac{-2\cos x \cdot \sin x}{1} = 0$

Thus $K = 0$

8) $\lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$

Solⁿ: Let $K = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \; \cdots \left(\frac{-\infty}{\infty}\right)$

Applying L' Hospital's rule

$K = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x \to 0} \frac{\sin x}{x} \tan x$

$= -\lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \tan x = -1 \cdot 0 = 0$

Thus $\underline{\underline{K = 0}}$

⟹ Evaluate the following limits.

9) $\lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$

Sol^n: let $K = \lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$ --- $(\infty - \infty)$

we need to simplify the given expression.

i.e $= \lim_{x \to 1} \left[\frac{x \log x - (x-1)}{(x-1)\log x}\right]$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospitals rule,

$K = \lim_{x \to 1} \left[\frac{x.(1/x) + \log x - 1}{(x-1)(1/x) + \log x}\right]$

$= \lim_{x \to 1} \left[\frac{x \log x}{(x-1) + x \log x}\right]$ --- $\left(\frac{0}{0}\right)$

$\therefore K = \lim_{x \to 1} \frac{1 + \log x}{1 + (1 + \log x)} = \frac{1+0}{2+0} = \frac{1}{2}$

Thus $K = 1/2$

10) $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right]$ --- $\cancel{(\infty - \infty)}$

i.e let $K = \lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right]$ --- $(\infty - \infty)$

$= \lim_{x \to 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)}\right]$ --- $\left(\frac{0}{0}\right)$

Applying L Hospitals rule

$= \lim_{x \to 2} \left[\frac{(1/x-1) - 1}{(x-2/x-1) + \log(x-1)}\right]$

we shall simplify

i.e $= \lim_{x \to 2} \left[\frac{2-x}{(x-2) + (x-1)\log(x-1)}\right]$ --- $\left(\frac{0}{0}\right)$

$K = \lim_{x \to 2} \left[\frac{-1}{1 + 1 + \log(x-1)}\right] = \frac{-1}{2}$

Thus $K = -1/2$

11) $\lim_{x \to 0} \left[\frac{a}{x} = \cot(x/a)\right]$

Sol^n: let $K = \lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right]$ --- $(\infty - \infty)$

i.e $= \lim_{x \to 0} \left[\frac{a}{x} - \frac{\cos(x/a)}{\sin(x/a)}\right]$

$= \lim_{x \to 0} \left[\frac{a \sin(x/a) - x\cos(x/a)}{x \sin(x/a)}\right]$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule.

$K = \lim_{x \to 0} \left[\frac{a \cdot 1/a \cdot \cos(x/a) + x \cdot 1/a \cdot \sin(x/a) - \cos(x/a)}{x \cdot 1/a \cdot \cos(x/a) + \sin(x/a)}\right]$

$= \lim_{x \to 0} \left[\frac{x \sin(x/a)}{x\cos(x/a) + a\sin(x/a)}\right]$ --- $\left(\frac{0}{0}\right)$

$K = \lim_{x \to 0} \left[\frac{x \cdot 1/a \cdot \cos(x/a) + \sin(x/a)}{x \cdot -1/a \cdot \sin(x/a) + \cos(x/a) + \cos(x/a)}\right]$

$K = \frac{0+0}{0+1+1} = \frac{0}{2} = 0$

Thus, $K = 0$

12) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2}\right]$

Sol^n: let $K = \lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2}\right]$ --- $\left(\infty - \frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{x - \log(1+x)}{x^2}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule.

$$K = \lim_{x\to 0} \frac{1-(1/1+x)}{2x} \cdots \left(\frac{0}{0}\right)$$

$$K = \lim_{x\to 0} \frac{1/(1+x)^2}{2} = \frac{1}{2}$$

Thus $K = 1/2$

13) $\lim_{x\to\pi/2} [2x\tan x - \pi\sec x]$

sol^n: let $K = \lim_{x\to\pi/2} [2x\tan x - \pi\sec x] \cdots (\infty-\infty)$

i.e $= \lim_{x\to\pi/2} \left[\frac{2x\cdot\sin x}{\cos x} - \pi\cdot\frac{1}{\cos x}\right]$

i.e $= \lim_{x\to\pi/2} \left[\frac{2x\sin x - \pi}{\cos x}\right] \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$$K = \lim_{x\to\pi/2} \left[\frac{2x\cos x + 2\sin x}{-\sin x}\right] = -2$$

Thus $K = -2$.

14) $\lim_{x\to 1} (1-x^2)\tan(\pi x/2) \cdots \cancel{(0\times\infty)}$

sol^n: let $K = \lim_{x\to 1} (1-x^2)\tan\left(\frac{\pi x}{2}\right) \cdots (0\times\infty)$

i.e $= \lim_{x\to 1} \frac{(1-x^2)}{\cot(\pi x/2)} \cdots \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$K = \lim_{x\to 1} \frac{-2x}{-(\pi/2)\operatorname{cosec}^2(\pi x/2)} = \frac{4}{\pi}$$

Thus $K = 4/\pi$

15) $\lim_{x\to 0} x\log x$

sol^n: let $K = \lim_{x\to 0} x\log x \cdots (0\times-\infty)$

i.e $= \lim_{x\to 0} \frac{\log x}{(1/x)} \cdots \left(\frac{-\infty}{\infty}\right)$

Now applying L' Hospital's rule.

$$K = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$$

Thus $K = 0$

16) $\lim_{x\to a} \log[2-(x/a)]\cot(x-a)$

sol^n: let $K = \lim_{x\to a} \log[2-(x/a)]\cot(x-a) \cdots (0\times\infty)$

i.e $= \lim_{x\to a} \frac{\log[2-(x/a)]}{\tan(x-a)} \cdots \left(\frac{0}{0}\right)$

Now appling L' Hospital's rule.

$$K = \lim_{x\to a} \frac{\frac{1}{[2-(x/a)]}(-1/a)}{\sec^2(x-a)}$$

Thus $K = -1/a$.

Type - 2.

The given expression or its simplified form will be in the 0/0 form ~~th~~ when $x=0$ or as $x\to 0$ but will involve terms of the form $x^2\sin x$, $x\sin^3 x$, $x\tan^2 x$ etc

In the event of applying the rule, the differentiation becomes tedious and we should not venture to do so. We can conveniently modify such terms so as to involve $(\sin x/x)^k$ or $(\tan x/x)^k$ or $(x/\sin x)^k$ or $(x/\tan x)^k$ which can be seperated out from the given expression. These terms become 1 as $x \to 0$ with the result we will be left with a simple expression (products get eliminated) in the 0/0 form for the application of L'Hospital's rule simplification at each step has to be explored.

Evaluate the following limits.

17) $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$

Sol^n: Let $K = \lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$ --- $\left(\frac{0}{0}\right)$

i.e $= \lim_{x \to 0} \frac{\tan x - x}{\frac{x^2 \cdot \tan x \cdot x}{x}} = \lim_{x \to 0} \frac{\tan x - x}{x^3} \lim_{x \to 0} \frac{x}{\tan x}$

Hence $K = \lim_{x \to 0} \frac{\tan x - x}{x^3} \cdot 1$ --- $\left(\frac{0}{0}\right)$

Applying L Hospital's rule.
$K = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{\tan^2 x}{3x^2}$

i.e $= \frac{1}{3} \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 = \frac{1}{3} \cdot 1 = \frac{1}{3}$

Thus $K = 1/3$

18) $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

Sol^n: Let $K = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$ --- $\left(\frac{0}{0}\right)$

i.e $= \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{\frac{x \cdot \sin^3 x \cdot x^3}{x^3}}$

$= \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^3$

Hence, $K = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot 1$ --- $\left(\frac{0}{0}\right)$

Applying L-Hospital's rule.
$K = \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3}$ --- $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{2 - 2\cos x}{12x^2}$ --- $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{2\sin x}{24x} = \frac{1}{12} \lim_{x \to 0} \frac{\sin x}{x}$

$= \frac{1}{12} \cdot 1 = \frac{1}{12}$.

Thus $K = \frac{1}{12}$

19) $\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$.

Sol^n: Let $K = \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$ --- $\left(\frac{0}{0}\right)$

i.e $= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \cdot \frac{\sin x}{x} \cdot x}$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \lim_{x\to 0} \frac{x}{\sin x}$$

$$K = \lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1 \; \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule

$$K = \lim_{x\to 0} \frac{e^x + e^{-x} - (2/1+x)}{2x} \cdots \left(\frac{0}{0}\right)$$

$$K = \lim_{x\to 0} \frac{e^x - e^{-x} + 2/(1+x)^2}{2} = 1$$

Thus $K = 1$.

20) $\lim_{x\to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x\tan^2 x}$

sol: let $K = \lim_{x\to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x\tan^2 x} \cdots \left(\dfrac{0}{0}\right)$

i.e $= \lim_{x\to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3} \cdot \lim_{x\to 0}\left(\dfrac{x}{\tan x}\right)^2$

$$= \lim_{x\to 0} \frac{1+\sin x - \cos x + \log(1-x)}{x^3} \cdot 1 \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule

$$K = \lim_{x\to 0} \frac{\cos x + \sin x - 1/1-x}{3x^2} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x + \cos x - 1/(1-x)^2}{6x} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\cos x - \sin x - 2/(1-x)^3}{6}$$

$$= \frac{-3}{6} = \frac{-1}{2}$$

Thus $K = -1/2$.

21) $\lim_{x\to 0}\left[\dfrac{1}{x^2} - \cot^2 x\right]$

sol: let $K = \lim_{x\to 0}\left[\dfrac{1}{x^2} - \cot^2 x\right] \cdots (\infty - \infty)$

i.e $= \lim_{x\to 0}\left[\dfrac{1}{x^2} - \dfrac{1}{\tan^2 x}\right] = \lim_{x\to 0}\left[\dfrac{\tan^2 x - x^2}{x^2\tan^2 x}\right]$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{x^4} \cdot \lim_{x\to 0}\left(\frac{x}{\tan x}\right)^2$$

$$K = \lim_{x\to 0} \frac{\tan^2 x - x^2}{x^4} \cdot 1 \cdots \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule.

$$K = \lim_{x\to 0} \frac{2\tan x\sec^2 x - 2x}{4x^3} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\tan x \cdot 2\sec^2 x\tan x + 2\sec^4 x - 2}{12x^2}$$

$$= 2\lim_{x\to 0} \frac{2\sec^2 x \cdot \tan^2 x + \sec^4 x - 1}{12x^2} \cdots \left(\frac{0}{0}\right)$$

Now $(\sec^4 x - 1) = (\sec^2 x - 1)(\sec^2 x + 1)$

i.e $(\sec^4 x - 1) = \tan^2 x(\sec^2 x + 1)$

$= \sec^2 x \cdot \tan^2 x + \tan^2 x$

Hence, $K = \lim_{x\to 0} \dfrac{2\sec^2 x \cdot \tan^2 x + \sec^2 x\tan^2 x + \tan^2 x}{6x^2}$

$$= \lim_{x\to 0} \frac{\tan^2 x(3\sec^2 x + 1)}{6x^2}$$

$$= \frac{1}{6}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2 \cdot \lim_{x\to 0}(3\sec^2 x + 1)$$

$$= \frac{1 \cdot 1 \cdot 4}{6} = \frac{2}{3}$$

$K = 2/3$

22) $\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$

Sol^n: Let $K=\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$ --- $(\infty-\infty)$

i.e $=\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x}$ --- $\left(\frac{0}{0}\right)$

$=\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^4}\cdot\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2$

$K=\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^4}\cdot 1$ ---- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$K=\lim_{x\to 0}\frac{2\sin x\cos x - 2x}{4x^3}$

$=\lim_{x\to 0}\frac{\sin 2x - 2x}{4x^3}$ --- $\left(\frac{0}{0}\right)$

$K=\lim_{x\to 0}\frac{2\cos 2x - 2}{12x^2}$ or $\lim_{x\to 0}\frac{\cos 2x - 1}{6x^2}$

$=\frac{1}{6}\lim_{x\to 0}\frac{-2\sin^2 x}{x^2}$

$=\frac{-1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=\frac{-1}{3}\cdot 1=\frac{-1}{3}$

Thus $K=-1/3$.

Type - 3.

Indeterminate forms : 0^0, ∞^0, 1^∞.

It is evident that the function involved will be of the form $[f(x)]^{g(x)}$ and ~~Taking logarithms on both side~~ ~~we ha~~

we have to find the limit as $x\to a$

Let $K=\lim_{x\to a}[f(x)]^{g(x)}$

Taking logarithms on both sides we have

$\log_e K=\lim_{x\to a} g(x)\cdot\log[f(x)]$

we can evaluate the limit on the RHS as already discussed and let us suppose that the limit is equal to l.

i.e $\log_e K = l \Rightarrow K=e^l$

which is the required limit.

Remark :$\Rightarrow$

why 1^∞ is indeterminate?

Let $K=\lim_{x\to a}[f(x)]^{g(x)}$ --- 1^∞

$\Rightarrow \log_e K=\lim_{x\to a} g(x)\cdot\log[f(x)]$ ... $\infty\times\log 1=\infty\times 0$

which is indeterminate. On the other hand if $K=\lim_{x\to a}[f(x)]^{g(x)}$ is of the form c^∞ where $c\neq 1$ we have

$\log_e K=\lim_{x\to a} g(x)\cdot\log[f(x)]=\infty\times\log c=\infty$

$\Rightarrow$ Evaluate the following limits.

23) $\lim_{x\to 1} x^{1/1-x}$

Sol^n: Let $K=\lim_{x\to 1} x^{1/1-x}$ --- 1^∞

$$\Rightarrow \log_e k = \lim_{x\to 1} \frac{1}{1-x} . \log x$$

$$= \lim_{x\to 1} \frac{\log x}{1-x} \; --- \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to 1} \frac{1/x}{-1} = -1$$

i.e $\log_e k = -1$ Thus $k = e^{-1} = 1/e$

24) $\lim_{x\to 0} ((\cos x))^{1/x^2}$

Let $k = \lim_{x\to 0} ((\cos x))^{1/x^2} \; ---- (1^{\infty})$

$$\Rightarrow \log_e k = \lim_{x\to 0} \frac{\log(\cos x)}{x^2} \; --- \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to 0} \frac{-\sin x/\cos x}{2x}$$

$$= \frac{-1}{2} \lim_{x\to 0} \frac{\tan x}{x} = \frac{-1}{2} \cdot 1$$

i.e $\log_e k = \frac{-1}{2} \Rightarrow k = e^{-1/2} = \frac{1}{\sqrt{e}}$

Thus $k = 1/\sqrt{e}$.

~~25~~) $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

sol^n: Let $k = \lim_{x\to \pi/2} (\sin x)^{\tan x} \; --- (1^{\infty})$

$$\Rightarrow \log_e k = \lim_{x\to \pi/2} \tan x . \log(\sin x) \; --- (\infty \times 0)$$

i.e $= \lim_{x\to \pi/2} \frac{\log(\sin x)}{\cot x} \; --- \left(\frac{0}{0}\right)$

Applying L' Hospital's rule

$$\log_e k = \lim_{x\to \pi/2} \frac{\cos x/\sin x}{-\mathrm{cosec}^2 x}$$

$$= \lim_{x\to \pi/2} -\sin x \cdot \cos x = 0$$

i.e $\log_e k = 0$ Thus, $k = e^0 = 1$

26) $\lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$

sol^n: Let $k = \lim_{x\to a} [2 - (x/a)]^{\tan(\pi x/2a)} \; --- (1^{\infty})$

$$\log_e k = \lim_{x\to a} \tan(\pi x/2a) \cdot \log[2-(x/a)] \; --- (\infty \times 0)$$

i.e $= \lim_{x\to a} \frac{\log[2-(x/a)]}{\cot(\pi x/2a)} \; ---- \left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to a} \frac{\frac{1}{[2-(x/a)]} \times \frac{-1}{a}}{-\mathrm{cosec}^2(\pi x/2a) \times \pi/2a} = \frac{2}{\pi}$$

i.e $\log_e k = \frac{2}{\pi}$ Thus $k = e^{2/\pi}$

Imp

~~27~~) $\lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x} \; ---$

sol^n: Let $k = \lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x} \; --- (1^{\infty})$

$$= \log_e k = \lim_{x\to 0} \frac{\log\{(a^x + b^x)/2\}}{x} \; -- \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to 0} \frac{\frac{2}{a^x+b^x}\cdot\frac{1}{2}(a^x \log a + b^x \log b)}{1}$$

$$= \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log(ab)^{1/2}$$

i.e $\log_e k = \log(ab)^{1/2} \Rightarrow k = (ab)^{1/2} = \sqrt{ab}$

Thus $k = \sqrt{ab}$

Note: Similar problems :→

Evaluate i) $\lim_{x\to 0}\left[\frac{a^x+b^x+c^x}{3}\right]^{1/x}$

ii) $\lim_{x\to 0}\left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$

proceeding on the same lines we can obtain the limits equal to $(abc)^{1/3}$ and $(abcd)^{1/4}$ respectively in the two problems.

28) $\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-x}\right)\right]^{\sec^2\left(\frac{\pi}{2-x}\right)}$

Sol^n: Let $k = \lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-x}\right)\right]^{\sec^2\left(\frac{\pi}{2-x}\right)}$ ---- 1^∞

put $y = \frac{\pi}{2-x}$ for convenience As $x\to 0, y\to \pi/2$

Hence, $k = \lim_{y\to\pi/2}(\sin^2 y)^{\sec^2 y}$ ---- 1^∞

$\log_e k = \lim_{y\to\pi/2} \sec^2 y \cdot \log(\sin^2 y)$ --- $(\infty\times 0)$

$\log_e k = \lim_{y\to\pi/2} \frac{\log(\sin^2 y)}{\cos^2 y}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$\log_e k = \lim_{y\to\pi/2} \frac{2\sin y\cos y/\sin^2 y}{-2\cos y\sin y} = -1$

i.e $\log_e k = -1$

Thus $k = e^{-1} = 1/e$.

29) $\lim_{x\to 1}(1-x^2)^{1/\log(1-x)}$

Sol^n: Let $k = \lim_{x\to 1}(1-x^2)^{1/\log(1-x)}$ --- (0^0)

$\Rightarrow \log_e k = \lim_{x\to 1}\frac{\log(1-x^2)}{\log(1-x)}$ --- $\left(\frac{-\infty}{-\infty}\right)$

Applying L' Hospital's rule

$\log_e k = \lim_{x\to 1}\frac{-2x/1-x^2}{-1/1-x}$

$= \lim_{x\to 1}\frac{2x(1-x)}{(1-x)(1+x)}$

i.e $\log_e k = \lim_{x\to 1}\frac{2x}{1+x} = 1$ Thus $k = e^1 = e$

30) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$

~~Sol^n: Let $k = \lim_{x\to\infty} x\log\left(1+\frac{a}{x}\right)$ --- $(\infty\times 0)$~~

Sol^n: Let $k = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$ --- (1^∞)

$\log_e k = \lim_{x\to\infty} x\log\left(1+\frac{a}{x}\right)$ --- $(\infty\times 0)$

$= \lim_{x\to\infty}\frac{\log(1+a/x)}{(1/x)}$ --- $\left(\frac{0}{0}\right)$

$$= \lim_{x\to\infty} \frac{1}{\left(1+\frac{a}{x}\right)} \frac{(-a/x^2)}{(-1/x^2)}$$

Applying L' Hospital's rule

$$\log_e k = \lim_{x\to\infty} \frac{\left[\frac{1}{\left(1+\frac{a}{x}\right)}\left(-\frac{a}{x^2}\right)\right]}{(-1/x^2)} = \lim_{x\to\infty} \frac{a}{\left(1+\frac{a}{x}\right)} = \frac{a}{1+0} = a$$

i.e $\log_e k = a$ Thus $k = e^a$

31) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Sol^n: Let $k = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$ --- (1^∞)

$\Rightarrow \log_e k = \lim_{x\to 0} \frac{\log(\sin x/x)}{x^2}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{\sin x/x} \cdot \frac{x\cos x - \sin x}{x^2}}{2x}$$

$$\log_e k = \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3}$$

$= 1 \cdot \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3}$ --- $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-x\sin x + \cancel{\cos x} - \cancel{\cos x}}{6x^2}$$

$$= \lim_{x\to 0} \frac{-1}{6} \frac{\sin x}{x} = \frac{-1 \cdot 1}{6} = \frac{-1}{6}$$

i.e $\log_e k = \frac{-1}{6}$ Thus $k = e^{-1/6}$

32) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Sol^n: Let $k = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$ --- 1^∞

$\Rightarrow \log_e k = \lim_{x\to 0} \frac{\log(\tan x/x)}{x}$ --- $\left(\frac{0}{0}\right)$

Applying L' Hospital's rule,

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{(\tan x/x)} \cdot \frac{x\sec^2 x - \tan x}{x^2}}{1}$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \cdot \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2}$$

$= 1 \cdot \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2}$ --- $\left(\frac{0}{0}\right)$

$$\log_e k = \lim_{x\to 0} \frac{x \cdot 2\sec^2 x \cdot \tan x + \sec^2 x - \sec^2 x}{2x}$$

$$= \lim_{x\to 0} \sec^2 x \tan x = 0$$

i.e $\log_e k = 0$ Thus $k = e^0 = 1$.

33) Find the value of the constants a and b such that $\lim_{x\to 0} \frac{a\cosh x - b\cos x}{x^2}$ may be equal to unity

Sol^n: Let $k = \lim_{x\to 0} \frac{a\cosh x - b\cos x}{x^2} = \frac{a-b}{0}$

We must have $a - b = 0$ in order to apply the L' Hospital's rule.

Hence, $k = \lim_{x\to 0} \frac{a\sinh x + b\sin x}{2x}$ -- $\left(\frac{0}{0}\right)$

$$k = \lim_{x\to 0} \frac{a\cosh x + b\cos x}{2} = \frac{a+b}{2}$$

But we must have $k=1$, by data

$\frac{(a+b)}{2} = 1$ or $(a+b) = 2$

By solving the Equations, $a-b=0$ and $a+b=2$ we get $a=1, b=1$ Thus

$a=1, b=1$

34> Find the constants a, b, c such that

$\lim_{x \to 0} \frac{a.e^{x} - b\cos x + c.e^{-x}}{x \sin x}$ may be equal to 2

Sol^n:- Let $k = \lim_{x\to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \sin x}$

i.e $= \lim_{x\to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x^2} . \lim_{x\to 0} \frac{x}{\sin x}$

$= \lim_{x\to 0} \frac{a.e^{x} - b\cos x + c.e^{-x}}{x^2} . 1$

$= \frac{a-b+c}{0}$

Therefore we must have

$a-b+c=0$ —— ①

with (1), we have 0/0 form.

Applying L'H Rule

$k = \lim_{x\to 0} \frac{ae^{x} + b\sin x - ce^{-x}}{2x} = \frac{a-c}{0}$

Again we must have $a-c=0$ to get 0/0 form that is $a=c$

Applying the rule again we have

$k = \lim_{x\to 0} \frac{ae^{x} + b\cos x + c.e^{-x}}{2}$

$= \frac{a+b+c}{2}$ and $k=2$ by data

i.e $\frac{(a+b+c)}{2} = 2$ and therefore

$a+b+c=4$ —— ③

Since $c=a$, ① and ③ becomes $2a-b=0$ and $2a+b=4$. By solving, we get $a=1$ and $b=2$. Hence $c=1$

Thus $a=1, b=2, c=1$

Partial Differentiation :⇒

$u = xy^2 + xy - x^2$

partial derivatives :⇒

$\frac{du}{dx}$

$\frac{\partial u}{\partial x} \to \frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x . \partial y}$

Let u be a function of two independent variables x and y.

That is $u = f(x,y)$

Recapitulating the defination of $\frac{dy}{dx}$ in the case of $y=f(x)$

The partial derivative of u w.r.to x and also w.r.to y are defined in a similar way

Here goes the definitions

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \quad \text{—— ①}$$

$$\frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \quad \text{—— ②}$$

Thus ① is called the partial derivative of u w.r.to x & ② is called the Partial derivative of u w.r.to y

It should be observed that in the Eq^n ① x is varying whereas y is remaining constant. Also in the Eq^n ② y is varying whereas x is remaining constant.

The derivative of u w.r.to x treating y as a constant is called as the partial derivative of u w.r.to x and is denoted by $\frac{\partial u}{\partial x}$ or u_x

Similarly the derivative of u w.r.to y treating x as a constant is called partial derivative of u w.r.to y and is denoted by $\frac{\partial u}{\partial y}$ or u_y.

$z = x^y + y^x$

Problems :→

1) If $u = x^3 - 3xy^2 + x + e^x\cos y + 1$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol^n :→ we have $u = x^3 - 3xy^2 + x + e^x\cos y + 1$

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x.\cos y$ (y-const)

Diff w.r.to x partially again

$\frac{\partial^2 u}{\partial x^2} = 6x + e^x\cos y$

Next, $\frac{\partial u}{\partial y} = -6xy - e^x\sin y$ ... (x-const)

Diff w.r.to y again

$\frac{\partial^2 u}{\partial y^2} = -6x - e^x\cos y$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x\cos y - 6x - e^x\cos y = 0$

Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2) If $u = e^{-2\pi^2 t}\sin\pi x.\sin\pi y$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Sol^n :→ we have $u = e^{-2\pi^2 t}\sin\pi x.\sin\pi y$

$\therefore \frac{\partial u}{\partial x} = e^{-2\pi^2 t}(\pi\cos\pi x).\sin\pi y$ (t & y are constant)

Diff this w.r.to x partially again

$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t}(-\pi^2\sin\pi x)\sin\pi y = -\pi^2 u$

$\frac{\partial u}{\partial y} = e^{-2\pi^2 t}\sin\pi x(\pi\cos\pi y)$ (t & y const)

Diff w.r.to y partially again

$\frac{\partial^2 u}{\partial y^2} = e^{-2\pi^2 t}.\sin\pi x(-\pi^2\sin\pi y) = -\pi^2 u$

Hence, LHS $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi^2 u - \pi^2 u = -2\pi^2 u$ ——— ①

Also, $\frac{\partial u}{\partial t} = e^{-2\pi^2 t}(-2\pi^2)\sin\pi x.\sin\pi y$ (x & y are treated const)

i.e RHS $= \frac{\partial u}{\partial t} = -2\pi^2 u$

Thus from ① & ② $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

3> If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ show that $xu_x + yu_y = 1$

Sol^n: $u = \log(x^2+y^2) - \log(x+y)$

$$u_x = \frac{1}{x^2+y^2}\cdot 2x - \frac{1}{x+y}\cdot 1$$

and $u_y = \frac{1}{x^2+y^2}\cdot 2y - \frac{1}{x+y}\cdot 1$

Now, $xu_x + yu_y = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} + \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$

$$= \frac{2(x^2+y^2)}{x^2+y^2} - \frac{(x+y)}{x+y}$$

$$= 2 - 1 = 1$$

Thus $x\cdot u_x + yu_y = 1$

4> If $u = \tan^{-1}\left(\frac{y}{x}\right)$ verify that $\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\cdot\partial y}$

Sol^n: we have $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

Now, $\frac{\partial^2 u}{\partial y\cdot\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

i.e. $= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$

$$= \frac{(x^2+y^2)(-1)-(-y)2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} \quad \text{---- (by quotient rule)}$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{---- (1)}$$

Also, $\frac{\partial^2 u}{\partial x\cdot\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\cdot\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{---- (2)}$$

Thus from (1) & (2) $\frac{\partial^2 u}{\partial y\cdot\partial x} = \frac{\partial^2 u}{\partial x\cdot\partial y}$

5> If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ show that

$$\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Sol^n: we have $\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

By data, $z = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$

$$\frac{\partial z}{\partial y} = x^2\cdot\frac{1}{1+(y/x)^2}\cdot\frac{1}{x} - \left\{y^2\cdot\frac{1}{1+(x/y)^2}\cdot\left(\frac{-x}{y^2}\right) + 2y\tan^{-1}\left(\frac{x}{y}\right)\right\}$$

i.e $\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}(x/y)$

$\frac{\partial z}{\partial y} = \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y\tan^{-1}(x/y)$

$\frac{\partial z}{\partial y} = x - 2y\tan^{-1}(x/y)$

Now diff w.r.to x partially we have

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\left[\frac{1}{1+(x/y)^2}\cdot\frac{1}{y}\right]$

or $\frac{\partial^2 z}{\partial x\cdot\partial y} = 1 - \frac{2y^2}{x^2+y^2}$

$= \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

Thus $\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{x^2-y^2}{x^2+y^2}$

6) If $z = \tan(y+ax) + (y-ax)^{3/2}$ show that $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$

sol^n: we have $z = \tan(y+ax) + (y-ax)^{3/2}$

$\frac{\partial z}{\partial x} = \sec^2(y+ax).a + \frac{3}{2}(y-ax)^{1/2}(-a)$

Diff this w.r.to x again partially

$\frac{\partial^2 z}{\partial x^2} = 2\sec(y+ax).\sec(y+ax).\tan(y+ax)a^2 + \frac{3}{2}.\frac{1}{2}(y-ax)^{-1/2}.a^2$

$\frac{\partial^2 z}{\partial x^2} = a^2\left\{2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right\}$ —— (1)

Now, $\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$

Also $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2\sec(y+ax).\sec(y+ax)\tan(y+ax) + \frac{3}{2}.\frac{1}{2}(y-ax)^{-1/2}$

$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}$ —— (2)

Using (2) in (1) we get

$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$

7) If $v = e^{a\theta}\cos(a\log r)$ Prove that $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

sol^n:→ $\frac{\partial v}{\partial r} = -e^{a\theta}.\sin(a\log r).\frac{a}{r}$ —— (1)

$\therefore \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\right) = -e^{a\theta}\left\{\sin(a\log r)\left(\frac{-a}{r^2}\right) + \cos(a\log r)\frac{a^2}{r^2}\right\}$

i.e $\frac{\partial^2 v}{\partial r^2} = \frac{a}{r^2}e^{a\theta}.\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r)$ —— (2)

Next, $\frac{\partial v}{\partial \theta} = a.e^{a\theta}\cos(a\log r)$

$\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial\theta}\right) = \frac{\partial^2 v}{\partial\theta^2} = a^2.e^{a\theta}\cos(a\log r)$ —— (3)

consider, $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}$ by using (1) (2) & (3)

$= \left\{\frac{a}{r^2}e^{a\theta}\sin(a\log r) - \frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\} + \left\{-\frac{a}{r^2}e^{a\theta}\sin(a\log r)\right\} + \left\{\frac{a^2}{r^2}e^{a\theta}\cos(a\log r)\right\}$

$= 0$

Hence the proof

8) If $u = \frac{1}{r}[f(r-at) + \phi(r+at)]$ show that

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

Soln: we have $u = \frac{1}{r}[f(r-at) + \phi(r+at)]$

$$\therefore \frac{\partial u}{\partial t} = \frac{1}{r}[f'(r-at)(-a) + \phi'(r+at)\cdot a]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{1}{r}[f''(r-at)\cdot a^2 + \phi''(r+at)a^2]$$

i.e $$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r}[f''(r-at) + \phi''(r+at)] \quad \text{---} (1)$$

Next, $$\frac{\partial u}{\partial r} = \frac{1}{r}[f'(r-at)\cdot 1 + \phi'(r+at)\cdot 1] + [f(r-at) + \phi(r+at)](-1/r^2)$$

$$\therefore r^2 \cdot \frac{\partial u}{\partial r} = r[f'(r-at) + \phi'(r+at)] - [f(r-at) + \phi(r+at)]$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = r[f''(r-at) + \phi''(r+at)] + [f'(r-at) + \phi'(r+at)] - [f'(r-at) + \phi'(r+at)]$$

i.e $$\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial u}{\partial r}\right) = r[f''(r-at) + \phi''(r+at)]$$

$$\therefore \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial u}{\partial r}\right) = \frac{a^2}{r}[f''(r-at) + \phi''(r+at)] \quad \text{---} (2)$$

Thus from (1) & (2).

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial u}{\partial r}\right)$$

9) If $u = e^{ax+by} f(ax-by)$ P.T $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$

Soln: $u = e^{ax+by} f(ax-by)$ by data

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + a\cdot e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial x} = a\cdot e^{ax+by} f'(ax-by) + au \quad \text{---} (1)$$

Next, $$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + b\, e^{ax+by} f(ax-by)$$

or $$\frac{\partial u}{\partial y} = -b\cdot e^{ax+by} f'(ax-by) + bu \quad \text{---} (2)$$

Now consider, LHS $= b\cdot\frac{\partial u}{\partial x} + a\cdot\frac{\partial u}{\partial y}$ by using (1) & (2)

$$\text{LHS} = b\{a e^{ax+by} f'(ax-by) + au\} + a\{-b e^{ax+by} f'(ax-by) + bu\}$$

$$= ab e^{ax+by} f'(ax-by) + abu - ab e^{ax+by} f'(ax-by) + abu$$

LHS $= 2abu$ = RHS.

Thus $$b\cdot\frac{\partial u}{\partial x} + a\cdot\frac{\partial u}{\partial y} = 2abu$$

10) Symmetric functions

If $u = \log\sqrt{x^2+y^2+z^2}$ S.T $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

Soln: ⇒ By data

$$u = \log\sqrt{x^2+y^2+z^2} = \frac{1}{2}\log(x^2+y^2+z^2)$$

The given u is symmetric funn of x, y, z. It is enough if we compute only one of the required partial derivative.

$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

$$= \frac{(x^2+y^2+z^2)(1) - x\cdot 2x}{(x^2+y^2+z^2)^2}$$

$$= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \quad \text{——} \; (1)$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$ —— (2)

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad \text{——} \; (3)$$

Adding (1) & (2), (3) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Thus $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

11) If $u = \log(x^3+y^3+z^3-3xyz)$ then Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence

$$\left\{ \text{S.T} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2} \right\}$$

Sol^n: Given $u = \log(x^3+y^3+z^3-3xyz)$ is a symmetric function

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{——} \; (1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} \quad \text{——} \; (2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \text{——} \; (3)$$

Adding (1), (2) and (3) we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)}$$

By std Elementary result

$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

Thus, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

12) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Sol^n: $u = (x^2+y^2+z^2)^{-1/2}$ is a symmetric fun^n of x, y, z.

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x$$

$$= -(x^2+y^2+z^2)^{-3/2}\cdot x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\{(x^2+y^2+z^2)^{-3/2}\cdot x\}$$

$$= -\{(x^2+y^2+z^2)^{-3/2}\cdot 1 + x(-3/2)(x^2+y^2+z^2)^{-5/2}\cdot 2x\}$$

$$= -\{(x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}\}$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (3)}$$

Adding (1) (2) and (3) we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$
$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$
$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

13) If $z(x+y) = x^2+y^2$ S.T $\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$

Sol^n: $z = \frac{x^2+y^2}{x+y}$ by data

is a symmetric fun^n of x, y

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad \text{--- (1)}$$

Similarly $\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$ --- (2)

Now, $$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{(x^2+2xy-y^2)-(y^2+2xy-x^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2-y^2)}{(x+y)^2} = \frac{2(x-y)(x+y)}{(x+y)^2}$$
$$= \frac{2(x-y)}{(x+y)}$$

$$\therefore \left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = \frac{4(x-y)^2}{(x+y)^2} \quad \text{--- (5)}$$

Now, $$4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2 - \cancel{x^2} - 2xy + \cancel{y^2} - \cancel{y^2} - 2xy + \cancel{x^2}}{(x+y)^2}\right]$$

i.e $$4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]$$
$$= \frac{4(x-y)^2}{(x+y)^2} \quad \text{--- (4)}$$

Thus from (3) and (4)

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

Note :⇒

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \quad \text{but} \quad \frac{\partial u}{\partial x} \neq \frac{1}{\left(\frac{\partial x}{\partial u}\right)} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq \frac{1}{\left(\frac{\partial y}{\partial u}\right)} \text{ etc}$$

14) If $x^x y^y z^z = c$. S.T $\frac{\partial^2 z}{\partial x \cdot \partial y} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$

and hence deduce that $\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \log_e x)^{-1}$

when $x = y = z$

Sol^n: we have $x^x y^y z^z = c$ and we have to treat z as a function of x and y in order to find the required partial derivative.

Taking logarithms on both sides of the given equation we get

$$x\log x + y\log y + z\log z = \log c$$

Diff partially w.r.to y bearing in the mind that z is a function of x and y we get

$$0 + \left(y\cdot\frac{1}{y} + 1\cdot\log y\right) + \left(z\cdot\frac{1}{z} + 1\cdot\log z\right)\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)}$$

Now, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-\partial}{\partial x}\left\{\frac{(1+\log y)}{(1+\log z)}\right\}$

i.e $\frac{\partial^2 z}{\partial x\cdot\partial y} = \left\{\frac{(1+\log z)\cdot 0 - (1+\log y)\cdot\frac{1}{z}\frac{\partial z}{\partial x}}{(1+\log z)^2}\right\}$ —— ①

Taking a note that the given function is symmetric we can write.

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)} \quad \text{—— ②}$$

using ② in ① we get

$$\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

Further when $x=y=z$, the RHS of the earlier expression on replacing y and z by x assumes the form.

$$\frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{-(1+\log x)(1+\log x)}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$= \frac{-1}{x(\log e + \log x)}$$

$$= \frac{-1}{x\log ex} = -(x\log ex)^{-1}$$

Thus, $\left[\frac{\partial^2 z}{\partial x\partial y}\right] = -(x\log ex)^{-1}$

15) If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ S.T $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

Soln :- $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ is a symmetric function.

Diff w.r to x b.s we have

$$2r\cdot\frac{\partial r}{\partial x} = 2(x-a) \text{ or } \frac{\partial r}{\partial x} = \frac{(x-a)}{r}$$

Further, $\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{(x-a)}{r}\right]$

Appling quotient rule we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{r\cdot 1 - (x-a)\cdot\frac{\partial r}{\partial x}}{r^2} = \frac{r - [(x-a)\cdot(x-a)/r]}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2 - (x-a)^2}{r^3}$$

Similarly, $\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - (y-b)^2}{r^3}$

$$\frac{\partial^2 r}{\partial z^2} = \frac{r^2 - (z-c)^2}{r^3}$$

Adding the results ① ② and ③ we get

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{1}{r^3}\left\{3r^2 - [(x-a)^2 + (y-b)^2 + (z-c)^2]\right\}$$

$$= \frac{1}{r^3}(3r^2 - r^2) = \frac{2r^2}{r^3} = \frac{2}{r}$$

Thus $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

16) If $x = r\cos\theta$ and $y = r\sin\theta$ prove that

(a). $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$

(b) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

soln :→ observing the result, we need to first express r as a function of x and y

$x = r\cos\theta$, $y = r\sin\theta$ gives $x^2+y^2 = r^2$

Now consider $r^2 = x^2+y^2$ which is a symmetric function

Diff w. r. to x partially

$2r \cdot \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now, $\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$

Applying quotient rule we have

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\cdot\frac{x}{r}}{r^2} = \frac{r^2-x^2}{r^3}$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{r^2-y^2}{r^3}$

$$LHS = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2-x^2}{r^3} + \frac{r^2-y^2}{r^3}$$

$$= \frac{1}{r^3}\left[2r^2 - (x^2+y^2)\right] = \frac{2r^2-r^2}{r^3} = \frac{1}{r}$$

Also, $RHS = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} = \frac{1}{r}\left\{\frac{x^2}{r^2} + \frac{y^2}{r^2}\right\}$

i.e $= \frac{1}{r}\left\{\frac{x^2+y^2}{r^2}\right\} = \frac{1}{r}\cdot\frac{r^2}{r^2} = \frac{1}{r}$

Thus LHS = RHS

(b) $\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$

since $x = r\cos\theta$ $\frac{\partial x}{\partial r} = \cos\theta$

$\therefore \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

17) If $u = f(r)$ where $r = \sqrt{x^2+y^2+z^2}$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$

soln: $u = f(r)$ where r is a funn of x, y, z

$\therefore \frac{\partial u}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$

But $r = \sqrt{x^2+y^2+z^2}$ gives $\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$

Hence $\frac{\partial u}{\partial x} = f'(r)\cdot\frac{x}{r}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left\{f'(r)\cdot\frac{x}{r}\right\}$

Applying product rule we have,

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r\cdot 1 - x\cdot\frac{\partial r}{\partial x}}{r^2}\right] + f''(r)\cdot\frac{\partial r}{\partial x}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\left[r - x\cdot\frac{x}{r}\right] + f''(r)\cdot\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\left[r - x\cdot\frac{x}{r}\right] + f''(r)\cdot\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r^2}\,\frac{(r^2-x^2)}{r} + f''(r)\cdot\frac{x^2}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3}(r^2-x^2) + f''(r)\cdot\frac{x^2}{r^2}$$

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{f'(r)}{r^3}(r^2-y^2) + f''(r)\cdot\dfrac{y^2}{r^2}$

$$\frac{\partial^2 u}{\partial z^2} = \frac{f'(r)}{r^3}(r^2-z^2) + f''(r)\cdot\frac{z^2}{r^2}$$

Now, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

$$= \frac{f'(r)}{r^3}\{(r^2-x^2)+(r^2-y^2)+(r^2-z^2)\} + \frac{f''(r)}{r^2}(x^2+y^2+z^2)$$

$$= \frac{f'(r)}{r^3}\{3r^2-(x^2+y^2+z^2)\} + \frac{f''(r)}{r^2}\cdot r^2 \quad (x^2+y^2+z^2=r^2)$$

$$= \frac{f'(r)}{r^3}2r^2 + f''(r) = \frac{2}{r}f'(r) + f''(r)$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$

Total derivative - Differentiation of Composite functions

If $u = f(x,y)$ then the total differential of u is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{——— (1)}$$

If $u = f(x,y)$ where x and y are functions of the independent variable t then u is said to be a composite function of the single variable t.

Also if $u = f(x,y)$ where both x and y are functions of two independent variables r, s then u is said to be a composite function of the ~~single~~ two variables ~~t~~ r and s.

The principle of differentiation of composite funⁿ is very much similar to that of the function of a funⁿ rule associated with the ordinary derivative of a function of a single independent variable.

We discuss two types involving partial derivatives

Type i:

Total derivative rule.

If $u = f(x,y)$ where $x = x(t)$ and $y = y(t)$ then u is a composite funⁿ of the single variable t. Therefore in principle we should be able to differentiate u with respect to t which is an ordinary derivative.

Thus we have with reference to (1).

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{——— (2)}$$

This is called as the total derivative of u

Type ii: Chain rule

If $u = f(x,y)$ where $x = x(r,s)$ and $y = y(r,s)$ then u is a composite funⁿ of two independent variables r, s. Therefore in principle we should be able to differentiate u w.r.to r & s which is ~~an ordinary~~ derivative partially.

Thus we have ~~with~~ reference to ① the following chain rule for the two partial derivatives.

It is convenient to write the rule having the data analysed in the following format.

$u \to (x,y) \to (r,s) \Rightarrow u \to (r,s)$ $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial s}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \; ; \; \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \text{--- ③}$$

Note :⟹ 1) The rules (2) and (3) can be established from the basic limit form definition of a partial derivative.

2) The rules (2) and (3) can be extended to functions involving more than two independent variables

3) The rules (2) and (3) can be successively applied for getting higher order derivatives of the given funⁿ.

4) The symbol $\to$ is used only to indicate the composition of the variables so that the associated rule can be written conveni-ently.

Find the total derivative of the following funⁿ and also verify the result by direct substitution.

1) $z = xy^2 + x^2y$ where $x = at$, $y = 2at$

solⁿ :⟹ $z = xy^2 + x^2y$: $x = at$, $y = 2at$

$\{z \to (x,y) \to t\} \Rightarrow z \to t$ & $\frac{dz}{dt}$ is a total derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)a + (2xy + x^2)2a$$

$$= (4a^2t^2 + 4a^2t^2)a + (4a^2t^2 + a^2t^2)2a$$

$$= (8a^2t^2)a + (5a^2t^2)2a = 8a^3t^2 + 10a^3t^2$$

$$= 18a^3t^2$$

∴ The total derivative $\frac{dz}{dt} = 18a^3t^2$ --- ①

Now, by direct substitution we have

$$z = xy^2 + x^2y = (at)(2at)^2 + (at)^2(2at)$$

$$= 4a^3t^3 + 2a^3t^3$$

$$= 6a^3t^3$$

i.e $z = 6a^3t^3$

Diff w.r.to t

$$\frac{dz}{dt} = 6a^3 \cdot 3t^2 = 18a^3t^2 \quad \text{--- ②}$$

Thus from ① & ② the result is verified