Module-1

Laplace Transform:-

* Definition of Laplace Transform:-

If $f(t)$ is a real valued function defined for all $t>0$ then the laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists, on integration of the indefinite integral we will be having a function of s and t. When this is evaluated bet^n the limits $t=0$ and $t=\infty$, we will be left with a function of 's' only and we shall denote it by $\bar{f}(s)$ where 's' is parameter, real or complex. Thus

$$L[f(t)] = \bar{f}(s)$$

Equivalently we can express this in the form.

$L^{-1}[\bar{f}(s)] = f(t)$ is called inverse laplace transform.

* Laplace transform of discontinuous functions from the basic definition

Working procedure for problems

step 1) Here $f(t)$ is composed of different functions in different subintervals of $(0,\infty)$

Suppose, $f(t) = \begin{cases} f_1(t), & 0<t<a \\ f_2(t), & a<t<b \\ f_3(t), & t>b \end{cases}$

then we consider the basic definition of $L[f(t)]$ that is $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

step2) By taking into account the intervals involved in the given $f(t)$ we express the integral in the equivalent form by using a property of definite integrals.

$$L[f(t)] = \int_0^a e^{-st} f(t)\,dt + \int_a^b e^{-st} f(t)\,dt + \int_b^\infty e^{-st} f(t)\,dt$$

step3) We evaluate the integrals to obtain $L[f(t)]$ as a function of s.

Illustrative Ex:-

1) Find $L[f(t)]$ where $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

soln:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$

using the relevant $f(t)$ in the integral we have

$$L[f(t)] = \int_0^4 e^{-st}\, t\,dt + \int_4^\infty e^{-st}\, 5\,dt$$

$$= \int_0^4 t\,e^{-st}\,dt + 5\int_4^\infty e^{-st}\,dt$$

using Bernoulli's rule for the first term in RHS we have,

$$L[f(t)] = \left[t\,\frac{e^{-st}}{-s}\right]_{t=0}^{4} - \left[1\cdot\frac{e^{-st}}{s^2}\right]_{t=0}^{4} + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= -\frac{1}{s}\left(4e^{-4s} - 0\right) - \frac{1}{s^2}\left(e^{-4s} - 1\right) - \frac{5}{s}\left(0 - e^{-4s}\right)$$

$$L[f(t)] = \frac{1}{s}e^{-4s} + \frac{1}{s^2}\left(1 - e^{-4s}\right)$$

Laplace transform of elementary functions:-

1) $L(a)$ where 'a' is a constant

$L(a) = \frac{a}{s}$ where $s>0$. If $a=1$ then

$L(1) = \frac{1}{s}$

2) $L(e^{at}) = \frac{1}{s-a}$

3) $L(\cosh at) = \frac{s}{s^2-a^2}$

4) $L(\sinh at) = \frac{a}{s^2-a^2}$ where $s>a$

5) $L(\cos at) = \frac{s}{s^2+a^2}$ where $s>0$

6) $L(\sin at) = \frac{a}{s^2+a^2}$ where $s>0$

7) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ where 'n' is a constant.

1) $L(a)$ where 'a' is constant

$$L(a) = \int_0^\infty e^{-st} a\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-a}{s}(0-1) = \frac{a}{s}$$

$(e^{-x} \to 0 \text{ as } x \to \infty)$

thus, $L(a) = \frac{a}{s}$ where $s > 0$. If $a = 1$ then $L(1) = \frac{1}{s}$

2) $L(e^{at})$

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$L(e^{at}) = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

thus, $L(e^{at}) = \frac{1}{s-a}$, where $s > a$

3) $L(\cosh at)$

$$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$ since $L(e^{at}) = \frac{1}{s-a}$

$$= \frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\} = \frac{1}{2}\cdot\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

thus, $L(\cosh at) = \frac{s}{s^2-a^2}$ where $s > a$

4) $$L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$

$$L(\sinh at) = \frac{a}{s^2-a^2}$$ where $s > a$

$\sqrt{n+1} = \lfloor n = n! \; (n>0)$

5) $L(\cos at)$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$$

using $\int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

$$L(\cos at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{+1}{s^2+a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{-1}{s^2+a^2}\left(0 - e^0(-s\cos 0 + a\sin 0)\right] = \frac{s}{s^2+a^2}$$

thus, $L(\cos at) = \frac{s}{s^2+a^2}$ where $s>0$

6) $L(\sin at)$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

using, $\int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$L(\sin at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$$L(\sin at) = \frac{-1}{s^2+a^2}\left[e^{-st}(s\sin at + a\cos at)\right]_0^\infty$$

$$= \frac{-1}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$$

thus, $L(\sin at) = \frac{a}{s^2+a^2}$ where $s>0$

7) $L(t^n)$

Defn of & properties of gamma functions is a pre-requisite

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

put $st = x$

$\therefore dt = \frac{dx}{s}$ and x varies from 0 to ∞

Now, $L(t^n) = \int_{x=0}^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$

$= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-x} x^n \, dx$

$= \frac{\Gamma(n+1)}{s^{n+1}}$

thus, $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ where 'n' is constant

8) $L(t^n)$ where 'n' is a positive integer.

$L(t^n) = \int_0^{\infty} e^{-st} t^n \, dt = \int_0^{\infty} t^n e^{-st} \, dt$

Integrating by parts we have,

$L(t^n) = \left[t^n \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} (nt^{n-1}) \, dt$

$= 0 + \frac{n}{s} \int_0^{\infty} e^{-st} t^{n-1} \, dt$

$L(t^n) = \frac{n}{s} L(t^{n-1})$

similarly $L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$,

$L(t^{n-2}) = \frac{n-2}{s} L(t^{n-3})$ etc

Also $L(t^2) = \frac{2}{s} L(t^2)$

$L(t^1) = \frac{1}{s} L(t^0)$

$L(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L(t^0)$

$= \frac{n!}{s^n} L(1) = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$

$L(t^n) = \frac{n!}{s^{n+1}}$ where 'n' is a positive integer.

Table of Laplace transform

	$f(t)$	$L[f(t)] = \bar{f}(s)$
1)	a	$\frac{a}{s}$
2)	e^{at}	$\frac{1}{s-a}$
3)	$\cosh at$	$\frac{s}{s^2-a^2}$
4)	$\cos at$	$\frac{s}{s^2+a^2}$
5)	$\sinh at$	$\frac{a}{s^2-a^2}$
6)	$\sin at$	$\frac{a}{s^2+a^2}$
7)	t^n	$\frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$
8)	t^n, $n=1,2,3\ldots$	$\frac{n!}{s^{n+1}}$

Observe the following illustrations based on the table of Laplace transform

1) $L(4) = \frac{4}{s}$

2) $L(e^{2t}) = \frac{1}{s-2}$

3) $L(e^{-t}) = \frac{1}{s+1}$

4) $L(3e^{4t}) = \frac{3}{s-4}$

5) $L(2\cosh 2t) = \frac{2s}{s^2-4}$

6) $L(3\sinh 2t) = \frac{6}{s^2-4}$

7) $L(\cos t) = \frac{s}{s^2+1}$

8) $L(3\cos 4t) = \frac{3s}{s^2+16}$

9) $L(t^5) = \frac{5!}{s^6} = \frac{120}{s^6}$

10) $L(4t^3) = \frac{4\times 3!}{s^4} = \frac{24}{s^4}$

11) $L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}}$

12) $L(\sin 2t) = \frac{2}{s^2+4}$

i.e. $L(\sqrt{t}) = \frac{1/2 \cdot \Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

WORKED PROBLEMS:-

* Find the laplace transform of the following functions

1) $\cosh^2 3t$

Soln:- Let $f(t) = \cosh^2 3t = \left[\frac{e^{3t} + e^{-3t}}{2}\right]^2$

$$f(t) = \frac{1}{4}(e^{6t} + e^{-6t} + 2)$$

$$L[f(t)] = \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$$

2) $e^{-2t}\sinh 4t$

2) Let $f(t) = e^{-2t}\sinh 4t$

$$= e^{-2t}\left(\frac{e^{4t} - e^{-4t}}{2}\right)$$

$$f(t) = \frac{1}{2}(e^{2t} - e^{-6t})$$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right] = \frac{4}{(s-2)(s+6)}$$

3) $\sin 5t \cos 2t$

Soln:- Let $f(t) = \sin 5t \cos 2t = \frac{1}{2}[\sin(5t+2t) + \sin(5t-2t)]$

$$f(t) = \frac{1}{2}(\sin 7t + \sin 3t)$$

$$L[f(t)] = \frac{1}{2}\left(\frac{7}{s^2+49} + \frac{3}{s^2+9}\right)$$

June 2018 4) $\cos t \cos 2t \cos 3t$

Soln:- Now, $\cos t \cos 2t = \frac{1}{2}(\cos(t+2t) + \cos(t-2t))$

$$= \frac{1}{2}(\cos 3t + \cos t)$$

$$\cos t \cos 2t \cos 3t = \frac{1}{2}[\cos 3t \cos 3t + \cos 3t \cos t]$$

$$f(t) = \frac{1}{2}\left\{\frac{1}{2}(\cos 6t + \cos 0) + \frac{1}{2}(\cos 4t + \cos 2t)\right\}$$

$f(t) = \frac{1}{4}(\cos 6t + 1 + \cos 4t + \cos 2t)$

$L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$

5) ie. $\sin^2(2t+1)$

Let, $f(t) = \sin^2(2t+1)$

$= \frac{1}{2}[1-\cos 2(2t+1)] = \frac{1}{2}(1-\cos(4t+2))$

$f(t) = \frac{1}{2}(1-\cos 4t . \cos 2 + \sin 4t \sin 2)$

$L[f(t)] = \frac{1}{2}\left\{\frac{1}{s} - \frac{s\cos 2}{s^2+16} + \frac{4\sin 2}{s^2+16}\right\}$

6) $(3t+4)^3 + 5^t$

soln:-

$f(t) = (27t^3 + 108t^2 + 144t + 64) + e^{\log 5 t}$

$L[f(t)] = 27.\frac{3!}{s^4} + 108.\frac{2!}{s^3} + 144.\frac{1!}{s^2} + \frac{64}{s} + \frac{1}{s-\log 5}$

$L[f(t)] = \cancel{27 \cdot 3!}\ \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s-\log 5}$

7) $3\sqrt{t} + \frac{4}{\sqrt{t}} = 3t^{1/2} + 4t^{-1/2}$

soln:-

$L[f(t)] = \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{4\Gamma(1/2)}{s^{1/2}} = \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{4\sqrt{\pi}}{\sqrt{s}}$

Here, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = 1/2 . \Gamma(1/2)$

$L[f(t)] = \sqrt{\frac{\pi}{s}}\left[\frac{3}{2s} + 4\right]$

8) $\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3 = t^{3/2} - \frac{1}{t^{3/2}} - 3\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)$

sol^n:- $f(t) = t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$$L[f(t)] = \frac{\Gamma(5/2)}{s^{5/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}} - \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{3\Gamma(1/2)}{s^{1/2}} \quad \text{—} \ (1)$$

$\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$, $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$

$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$ $\Gamma(n+1) = n\Gamma n$

$\Gamma n = \frac{n}{\Gamma n+1}$

substituting these values in (1) we get,

$$L[f(t)] = \sqrt{\pi}\left[\frac{3}{4s^2\sqrt{s}} + 2\sqrt{s} - \frac{3}{2s\sqrt{s}} + \frac{3}{\sqrt{s}}\right]$$

9) $t^{-3/2} + t^{5/2}$

$$L[f(t)] = \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}}$$

$$\Gamma(-3/2) = -\frac{2}{3}\frac{\Gamma(1/2)}{-1/2} = \frac{4\sqrt{\pi}}{3}$$

$$\Gamma(7/2) = \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$$

substituting these values in (1) we get

$$L[f(t)] = \frac{4\sqrt{\pi}}{3s^{-3/2}} + \frac{15}{8}\frac{\sqrt{\pi}}{s^{7/2}}$$

$$L[f(t)] = \sqrt{\pi}\left[\frac{4}{3}s^{3/2} + \frac{15}{8}\frac{1}{s^{7/2}}\right]$$

* Properties of Laplace transforms:-

1) If $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

proof:- We have definition

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \quad \text{---} (1)$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt = \int_0^\infty e^{-st+at} f(t)\,dt$$

comparing this integral with the integral in (1) being denoted by $\bar{f}(s)$ we observe that $(s-a)$ has replaced 's'.

Thus $L[e^{at} f(t)] = \bar{f}(s-a)$ → this property is called shifting property

2) If $L[f(t)] = \bar{f}(s)$, then

~~proof~~:- $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$ where 'n' is a positive integer

proof:- We establish the result by the principle of mathematical induction.

We have, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

diff. w.r.t 's' on both sides we have

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}] f(t)\,dt$$

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t) f(t)\,dt$$

$$(-1)\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}[t\, f(t)]\,dt = L[t\, f(t)]$$

This verifies the result for $n=1$

Let us assume the result to be true for $n=k$

$$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^k f(t)]\,dt$$

differentiating w.r.t 's' again we get.

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)]\,dt$$

$$= L[t\,f(t)] \quad \text{--- (1)}$$

This verifies the result for $n=1$

Let us assume the result to be true for $n=k$

i.e $(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = L[t^k f(t)]$ --- (2)

$$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^k f(t)\,dt]$$

diff. w.r.t 's' again we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)\,dt]$$

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t)[t^k f(t)]\,dt$$

Multiply by (-1) we get,

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^{k+1} f(t)]\,dt$$

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = L[t^{k+1} f(t)] \quad \text{--- (3)}$$

comparing (2) and (3) we conclude that the result is true for $n=k+1$. Hence by the principle of mathematical induction the result is true for all +ve integral values of n.

thus, $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n \bar{f}^n(s)$

This property is called the derivative of the transform property.

Property 3) If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

Proof:- We have, $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\,dt$

$$\int_s^{\infty} \bar{f}(s)\,ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^{\infty}\int_s^{\infty} e^{-st} f(t)\,ds\,dt$$, on changing the order of integration

$$= \int_0^{\infty}\left[\frac{e^{-st}}{-t}\right]_s^{\infty} f(t)\,dt = \int_0^{\infty}\left[0 - \frac{e^{-st}}{-t}\right] f(t)\,dt$$

$$\int_s^{\infty} \bar{f}(s)\,ds = \int_0^{\infty} e^{-st}\left[\frac{f(t)}{t}\right] dt = L\left[\frac{f(t)}{t}\right]$$

thus $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

Properties at a glance:-

If $L[f(t)] = \bar{f}(s)$ then we have,

1) $L[e^{at} f(t)] = \bar{f}(s-a)$

2) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

In particular, $L[t\, f(t)] = -\frac{d}{ds}[\bar{f}(s)]$,

$L[t^2 f(t)] = \frac{d^2}{ds^2}[\bar{f}(s)]$

3) $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

Ex :- Find the laplace transform of the following functions

1) $e^{-2t}(2\cos 5t - \sin 5t)$

Sol^n:- Let $f(t) = 2\cos 5t - \sin 5t$

$$L[f(t)] = 2 \cdot \frac{s}{s^2+5^2} - \frac{5}{s^2+25} = \frac{2s-5}{s^2+25}$$

Now, $L[e^{-2t} f(t)] = \left\{ \frac{2s-5}{s^2+25} \right\}_{s \to s+2}$

$$= \frac{2(s+2)-5}{(s+2)^2+25}$$

thus, $L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}$

2) $e^{-t}\cos^2 3t$

Sol^n:- Let $f(t) = \cos^2 3t = \frac{1+\cos 6t}{2}$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

Now, $L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s \to (s+1)}$

$$L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+36}\right]$$

3) $e^{-4t} t^{-5/2}$

Sol^n- $f(t) = t^{-5/2}$ (use 9th problem)

$$L[f(t)] = \frac{4\sqrt{\pi}}{3} s^{3/2}$$

Now, $L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}[s^{3/2}]_{s \to s+4}$

$$L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}(s+4)^{3/2}$$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$ 17

4) $(1+3t\,e^{2t})^2$

Sol^n:- $f(t) = (1+3t\,e^{2t})^2 = 1+6t\,e^{2t} + 9t^2 e^{4t}$

$L[f(t)] = L(1) + 6L(e^{2t}.t) + 9L(e^{4t}.t^2)$

$= \frac{1}{s} + 6\,L[t]\}_{s\to(s-2)} + 9\,L(t^2)_{s\to(s-4)}$

But, $L(t) = \frac{1}{s^2}$ and $L(t^2) = \frac{2}{s^3}$

$L(1+3t\,e^{2t})^2 = \frac{1}{s} + \frac{6}{(s-2)^2} + \frac{18}{(s-4)^3}$

5) $\sinh at \sin at$

Sol^n:- $f(t) = \frac{1}{2}(e^{at}\sin at - e^{-at}\sin at)$

$L[f(t)] = \frac{1}{2}\left\{L(\sin at)_{s\to s-a} - L(\sin at)_{s\to s+a}\right\}$

But, $L(\sin at) = \frac{a}{s^2+a^2}$

Hence, $L(\sinh at \sin at) = \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right\}$

$= \frac{a}{2}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$

$= \frac{a}{2}\left[\frac{4as}{(s^2+2a^2)^2-4a^2s^2}\right]$

$L(\sinh at \sin at) = \frac{2a^2 s}{s^4+4a^4}$

5) $\sin^3 2t \cosh t$

Sol^n- $f(t) = \sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$

$L(\sin^3 2t) = \frac{1}{4}\left(\frac{3\times 2}{s^2+4} - \frac{6}{s^2+36}\right) = \frac{48}{(s^2+4)(s^2+36)}$

Now, $L(\cosh t \sin^3 2t) = L\left[\frac{e^t+e^{-t}}{2}.\sin^3 2t\right]$

$$= \frac{1}{2}\left(L(e^{t}\sin^3 2t) + L(e^{-t}\sin^3 2t)\right)$$

$$= \frac{1}{2}\left\{L(\sin^3 2t)_{s \to s-1} + L(\sin^3 2t)_{s \to s+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{48}{[(s-1)^2+4][(s-1)^2+36]} + \frac{48}{[(s+1)^2+4][(s+1)^2+36]}\right\}$$

$$L(\cosh t \sin^3 2t) = 24\left\{\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right\}$$

6) $t\cos at$

Sol^n- $f(t) = \cos at \qquad L[f(t)] = \frac{s}{s^2+a^2}$

NOW, $L[t f(t)] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] = -\left\{\frac{(s^2+a^2)1 - s.2s}{(s^2+a^2)^2}\right\}$

$$L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

7) $t^2\sin at$

Sol^n- $L[t^2\sin at]$

$$L[t^2 f(t)] = \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right] = \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right\}$$

$$L[t^2\sin at] = \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

$$L[t^2\sin at] = \frac{(s^2+a^2)^2(-2a) + 2as.2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{2a(s^2+a^2)\{-(s^2+a^2)+4s^2\}}{(s^2+a^2)^4}$$

$$L[t^2\sin at] = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

8) $t^3 \sin t$

sol^n:- Let $f(t) = \sin t = \frac{1}{s^2+1}$

Now, $L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = \frac{-d}{ds} \cdot \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$

$L(t^3 \sin t) = \frac{-d}{ds}\frac{d}{ds}\left\{\frac{-2s}{(s^2+1)^2}\right\} = \frac{d}{ds}\left\{\frac{(s+1)^2 . 2 - 2s . 2(s^2+1) 2s}{(s^2+1)^4}\right\}$

$L(t^3 \sin t) = \frac{d}{ds}\left\{\frac{2(s^2+1)(s^2+1-4s^2)}{(s^2+1)^4}\right\}$

$= 2\frac{d}{ds}\left\{\frac{(1-3s^2)}{(s^2+1)^3}\right\}$

$L[t^3 \sin t] = 2\left\{\frac{(s^2+1)^3(-6s) - (1-3s^2) . 3(s^2+1)^2 . 2s}{(s^2+1)^6}\right\}$

$= -12s(s^2+1)^2\left\{\frac{(s^2+1)+(1-3s^2)}{(s^2+1)^6}\right\}$

$L[t^3 \sin t] = \frac{24s(s^2-1)}{(s^2+1)^4}$

9) $t^3 \cosh t$

sol^n:- $f(t) = t^3 \cosh t$

$f(t) = t^3\left(\frac{e^t+e^{-t}}{2}\right) = \frac{1}{2}\left\{e^t t^3 + e^{-t} t^3\right\}$

$L[f(t)] = \frac{1}{2}\left[L(t^3)_{s \to s-1} + L(t^3)_{s \to s+1}\right\}$

But $L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

$L[t^3 \cosh t] = \frac{1}{2}\left\{\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right\} = 3\left\{\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right\}$

10) $t^5 e^{4t} \cosh 3t = t^5 e^{4t} \cdot \frac{1}{2}(e^{3t} + e^{-3t})$

$f(t) = \frac{1}{2}(e^{7t} t^5 + e^t t^5)$

$= \frac{1}{2}\left[L(t^5)_{s \to s-7} + L(t^5)_{s \to s-1}\right\}$ But $L(t^5) = \frac{5!}{s^6} = \frac{120}{s^6}$

$L[t^5 e^{4t} \cosh 3t) = \frac{1}{2}\left\{\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right\} = 60\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$

11) $t e^{-2t} \sin 4t$

Sol^n:- $f(t) = t e^{-2t} \sin 4t$

$L(\sin 4t) = \frac{4}{s^2+16}$

$L(e^{-2t}\sin 4t) = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$

Hence, $L[t e^{-2t} \sin 4t] = \frac{-d}{ds}\left\{\frac{4}{s^2+4s+20}\right\} = \frac{4(2s+4)}{(s^2+4s+20)^2}$

$L[t e^{-2t} \sin 4t] = \frac{8(s+2)}{(s^2+4s+20)^2}$

11) ST $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Sol^n:- We have, $\int_0^\infty e^{-st} t^3 \sin t \, dt = L(t^3 \sin t)$

Reffering to problem for the RHS we have,

$\int_0^\infty e^{-st} t^3 \sin t \, dt = \frac{24s(s^2-1)}{(s^2+1)^4}$.

Thus by putting $s=1$ we get, $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0$

12) Show that $\int_0^\infty t e^{-2t} \sin 4t \, dt = \frac{1}{25}$

Sol^n:- We have, $\int_0^\infty e^{-st} t \sin 4t \, dt = L(t \sin 4t)$ —(1)

Now, $L(t \sin 4t) = \frac{-d}{ds} L(\sin 4t) = \frac{-d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{(s^2+16)^2}$

Hence (1) becomes $\int_0^\infty e^{-st} t \sin 4t \, dt = \frac{8s}{(s^2+16)^2}$

thus by putting $s=2$ we get,

$\int_0^\infty e^{-2t} t \sin 4t \, dt = \frac{16}{400} = \frac{1}{25}$

13) Find the value of $\int_0^\infty t\, e^{-3t} \cos 2t \, dt$ using Laplace transform.

Sol^n:- We have $\int_0^\infty e^{-st} t \cos 2t \, dt = L(t\cos 2t)$

$$L(t\cos 2t) = \frac{s^2-4}{(s^2+4)^2}$$

using this result in the RHS of (1) we have,

$$\int_0^\infty e^{-st} t \cos 2t \, dt = \frac{s^2-4}{(s^2+4)^2}$$

Thus by putting $s=3$ we get,

$$\int_0^\infty e^{-3t} t \cos 2t \, dt = \frac{5}{169}$$

* Find the laplace transform of the following functions.

* $\frac{1-e^{-at}}{t}$

Let $f(t) = 1-e^{-at}$

$$\bar{f}(s) = L[f(t)] = \frac{1}{s} - \frac{1}{s+a}$$

We have, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

Hence, $L\left[\frac{1-e^{-at}}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+a}\right) ds$

$$= \left[\log(s) - \log(s+a)\right]_s^\infty = \left[\log\left[\frac{s}{s+a}\right]\right]_s^\infty$$

$$= \lim_{s\to\infty}\left[\log\left(\frac{s}{s+a}\right)\right] - \log\left(\frac{s}{s+a}\right)$$

$$= L\left[\frac{1-e^{-at}}{t}\right]$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log 1 - \log\left(\frac{s}{s+a}\right)$$

$$= L\left[\left(\frac{1-e^{-st}}{t}\right)\right]$$

$$= \log\left(\frac{s+a}{s}\right)$$

2) $\frac{\cos at - \cos bt}{t}$

Sol^n:- Let, $f(t) = \cos at - \cos bt$

$$\bar{f}(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Hence, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\left(\frac{s^2+a^2}{s^2+b^2}\right)\right)\right]_s^\infty = \frac{1}{2}\lim_{s\to\infty}\left[\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

3) $\frac{\sinh t}{t}$

Sol^n:- Let $f(t) = \sinh t$

$$L\left[\frac{\sinh t}{t}\right] = \int_s^\infty \frac{1}{s^2-1}\, ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty}\left[\frac{1}{2}\log\left[\frac{s(1-1/s)}{s(1+1/s)}\right] - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right.$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s-1}{s+1}\right)\right\} = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

thus, $L\left[\frac{\sinh t}{t}\right] = \log\sqrt{\frac{s+1}{s-1}}$

4) $\frac{2\sin t \sin 5t}{t}$

Sol:- $f(t) = 2\sin t \sin 5t = 2\times\frac{1}{2}[\cos(-4t) - \cos 6t]$

$f(t) = \cos 4t - \cos 6t$

$$f(s) = \frac{s}{s^2+16} - \frac{s}{s^2+36}$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty\left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$$

$$= L\left[\frac{2\sin t\sin 5t}{t}\right] = \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]^\infty$$

$$= \lim_{s\to\infty}\left[\frac{1}{2}\log\left(\frac{s^2(1+16/s^2)}{s^2(1+36/s^2}\right]\right. - \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$$= L\left[\frac{2\sin t\sin 5t}{t}\right] = \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{2}\log\left[\frac{s^2+36}{s^2+16}\right]$$

$$= L\left[\frac{2\sin t\sin 5t}{t}\right]$$

$$= \log\sqrt{\frac{s^2+36}{s^2+16}}$$

5) $\frac{\sin^2 t}{t}$

Soln:- Let $f(t) = \sin^2 t$

$= \frac{1}{2}(1-\cos 2t)$

$\bar{f}(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

Hence $L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= -\frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$

Dec 2018 6) $\frac{\sin at}{t}$

Soln:- $f(t) = \sin at \quad \therefore \bar{f}(s) = \frac{a}{s^2+a^2}$

Hence, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds$

$L\left[\frac{\sin at}{t}\right] = a \cdot \frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$L\left(\frac{\sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}(s/a) = \cot^{-1}(s/a)$

* Evaluate the following integrals using Laplace transforms.

1) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$

Sol^n WKT $\int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t}\right]dt$

$= L\left[\frac{\cos 6t - \cos 4t}{t}\right]$

from problem

$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \log\left(\sqrt{(s^2+16)/(s^2+36)}\right)$

i.e $\int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t}\right]dt = \log\left(\sqrt{(s^2+16)/(s^2+36)}\right)$

thus by putting $s=0$ we get.

$\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right)dt = \log\sqrt{\frac{16}{36}} = \log\left(\frac{2}{3}\right)$

2) We shall first find $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

Sol^n- Now, $L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty [L(e^{-at}) - L(e^{-bt})]\,dt$

$= L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty [L(e^{-at}) - L(e^{-bt})]\,dt$

$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds = \log\left[\left(\frac{s+a}{s+b}\right)\right]_s^\infty$

$= \lim_{s\to\infty} \log\left[\frac{s(1+a/s)}{s(1+b/s)}\right] - \log\left(\frac{s+a}{s+b}\right)$

$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$

$= L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$

i.e $\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt$

$= \log\left(\frac{s+b}{s+a}\right)$

thus by putting $s=0$ we get,

$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log(b/a)$

3) $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

sol:- $L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty L(e^{-t}\sin t)\,ds$

$= \int_s^\infty L(\sin t)_{s\to s+1}\,ds$

$= \int_s^\infty \frac{1}{(s+1)^2+1}\,ds = \left[\tan^{-1}(s+1)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}(s+1)$

$L\left[\frac{e^{-t}\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s+1)$

$= \cot^{-1}(s+1)$

$\int_0^\infty e^{-st}\frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(s+1)$

Thus by putting $s=0$ we get, $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

$= \cot^{-1}(1) = \frac{\pi}{4}$

$L(\sin t) = \frac{1}{s^2+1}$

$\int_s^\infty \frac{1}{(s^2+1)}\,ds = \left[\tan^{-1}s\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}s$

$= \frac{\pi}{2} - \tan^{-1}s$

put $s=1$, $\frac{\pi}{2} - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

4) Find $L\left[\frac{\sin^2 t}{t^2}\right]$

Sol:- WKT $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Further we have, $L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \left[\int_s^\infty \bar{f}(s)\,ds\right] ds$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right]$$

$$L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{2}\log\left(\sqrt{\frac{s^2+4}{s}}\right) ds$$

$$= \int_s^\infty \frac{1}{2}\log\sqrt{\frac{s^2+4}{s^2}}\,ds$$

$$= \frac{1}{4}\int_s^\infty \log\left(1+\frac{4}{s^2}\right) ds = \frac{1}{4}\int_s^\infty \log\left[1+\frac{4}{s^2}\right]\cdot 1\,ds$$

Integrating by parts we have,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left\{\left[\log\left(1+\frac{4}{s^2}\right)\cdot s\right]_s^\infty - \int_s^\infty s\cdot\frac{1}{1+(4/s^2)}\left(\frac{-8}{s^3}\right)ds\right\}$$

$$= \frac{1}{4}\left\{\left[0-\log\left(1+\frac{4}{s^2}\right)s\right] + \int_s^\infty \frac{8}{s^2+4}\,ds\right\}$$

$$= -\frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) + \left[\tan^{-1}(s/2)\right]_s^\infty$$

$$= \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \left[\frac{\pi}{2} - \tan^{-1}(s/2)\right]$$

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}(s/2)$$