Mathematics - II

Vector Calculus.

* Vector differentiation :=>.

Scalar and vector fields :

If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a scalar point function and we say that a scalar field ϕ is defined in R.

Ex : 1. $\phi = x^2 + y^2 + z^2$ 2. $\phi = xy^2z^3$

If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called a 'vector point function' and we say that a 'vector field' $\vec{A}$ is defined in R.

Ex : 1. $\vec{A} = x^2 i + y^2 j + z^2 k$

2. $\vec{A} = xyz i + yz j + z k$.

Operators :

i) The vector differential operator ∇, read as 'Nabla' or 'Del' is defined by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$$

ii) The Laplacian operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$$

Gradient of a scalar field :

If $\phi(x, y, z)$ is continuously differentiable scalar function, then the 'gradient of ϕ' is defined to be $\nabla\phi$.

i.e. $\text{grad}\, \phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$ [vector quantity]

Divergence of a vector field :⇒

If $\vec{A}(x,y,z)$ is continuously differentiable vector function, then the 'divergence of $\vec{A}$' is defined to be $\nabla \cdot \vec{A}$.

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (a_1 i + a_2 j + a_3 k)$$

$$\text{i.e div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \text{ (scalar quantity)}$$

Curl of a Vector field :⇒

If $\vec{A}(x,y,z)$ is a continuously differentiable vector function then, 'curl of $\vec{A}$' is defined to be $\nabla \times \vec{A}$.

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{Curl. } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\text{Curl } \vec{A} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z} \right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

[vector quantity]

Note :⇒ If $\phi(x,y,z)$ is a continuously differentiable Scalar function, we define the Laplacian of ϕ as follows.

$$\text{Laplacian of } \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \text{ [scalar quantity]}$$

Obviously Laplacian of a scalar function is a Scalar quantity.

Examples :-

1> Given $\phi = x^2y + y^2z + z^2x$ Let us find $\nabla\phi$ and $\nabla^2\phi$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi = (2xy + z^2) i + (x^2 + 2yz) j + (y^2 + 2xz) k$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2x + 2y + 2z$$

2> Given $\vec{A} = x^2yz i + y^2zx j + z^2xy k$ Let us find div $\vec{A}$ and curl $\vec{A}$.

$$\text{div } \vec{A} = \nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2yz i + y^2zx j + z^2xy k)$$

$$\text{div } \vec{A} = \nabla\cdot\vec{A} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2zx) + \frac{\partial}{\partial z}(z^2xy)$$

$$\text{div } \vec{A} = \nabla\cdot\vec{A} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\text{curl } \vec{A} = \nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$$

$$= i(z^2x - y^2x) - j(z^2y - x^2y) + k(y^2z - x^2z)$$

$$= x(z^2 - y^2) i + y(x^2 - z^2) j + z(y^2 - x^2) k$$

Geometrical meaning of the gradient :

If $\phi(x, y, z)$ is a scalar function then grad ϕ is a vector normal to the surface $\phi(x, y, z) = c$

Note :

1> obviously the unit vector normal $\hat{n}$ along $\nabla\phi$ is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

2> The angle between the two surfaces is defined to be equal to the angle between their normals. If $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ be the eqns. of the two surfaces.

then, $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$, where θ is the angle between the normals.

If $\theta = \frac{\pi}{2}$ then the surfaces are said to intersect each other orthogonally.

when $\theta = \frac{\pi}{2}$, $\cos\theta = \cos\left(\frac{\pi}{2}\right) = 0 \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$

This is the condition for the surfaces to intersect at right angles.

Directional derivative :⇒

If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is a given direction, then $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \dfrac{\vec{d}}{|\vec{d}|}$ is called as the directional derivative of ϕ along $\hat{n}$.

* A vector $\vec{v}$ whose divergence is zero is called a "solenoidal vector."
* A vector function $\vec{v}(x, y, z)$ is said to be "irrotational" if curl $\vec{v} = \vec{0}$.

Note :⇒

1) $\nabla = \dfrac{\partial}{\partial x} i + \dfrac{\partial}{\partial y} j + \dfrac{\partial}{\partial z} k = \Sigma \dfrac{\partial}{\partial x} i$

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = \Sigma \dfrac{\partial^2}{\partial x^2}$

2) grad $\phi = \nabla\phi$: div $\vec{A} = \nabla \cdot \vec{A}$, curl $\vec{A} = \nabla \times \vec{A}$

$\nabla^2\phi = \nabla \cdot \nabla\phi$.

3) $\nabla\phi$ is a vector normal to the surface $\phi = (x, y, z) = c$ and $\dfrac{\nabla\phi}{|\nabla\phi|}$ is the unit vector normal to the surface.

4) Directional derivative of $\phi(x, y, z)$ along a given direction $\vec{D}$ is $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \dfrac{\vec{D}}{|\vec{D}|}$ and also directional derivative is maximum along $\nabla\phi$.

Problems :→

I> Find the unit vector normal to the following. surfaces at the indicated points.

i> $x^2y + 2xz = 4$ at $(2, -2, 3)$

sol^n : Let $\phi = x^2y + 2xz$, $\nabla\phi$ is a vector normal to the surface.

we have, $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$.

i.e $\nabla\phi = (2xy + 2z) i + x^2 j + 2x k$

$[\nabla\phi]_{(2,-2,3)} = -2i + 4j + 4k = 2(-i + 2j + 2k)$

The required unit vector normal, $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Thus, $\hat{n} = \frac{2(-i + 2j + 2k)}{\sqrt{2^2(1+4+4)}} = \boxed{\frac{-i + 2j + 2k}{3}}$

2> $xy^3z^2 = 4$ at $(-1, -1, 2)$

sol^n: Let $\phi = xy^3z^2$, $\nabla\phi$ is a vector normal to the surface

$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

i.e $\nabla\phi = y^3z^2 i + 3xy^2z^2 j + 2xy^3z k$

$[\nabla\phi]_{(-1,-1,2)} = -4i - 12j + 4k = -4(i + 3j - k)$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4(i + 3j - k)}{\sqrt{4^2(1+9+1)}} = \boxed{\frac{-(i + 3j - k)}{\sqrt{11}}}$

II> Find the directional derivative of the following

1> $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ along $2i - j - 2k$

sol^n : $\phi = x^2yz + 4xz^2$

$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$\nabla\phi = (2xyz + 4z^2) i + (x^2z) j + (x^2y + 8xz) k$

$[\nabla\phi]_{(1,-2,-1)} = 8i - j - 10k$

The unit vector in the direction of $2i - j - 2k$ is

$$\hat{n} = \frac{2i-j-2k}{\sqrt{4+1+4}} = \frac{2i-j-2k}{3}$$

The required directional derivative is.

$$\nabla\phi\cdot\hat{n} = (8i-j-10k)\cdot\frac{(2i-j-2k)}{3}$$

$$\text{Thus, } \nabla\phi\cdot\hat{n} = \frac{(8)(2)+(-1)(-1)+(-10)(-2)}{3} = \boxed{\frac{37}{3}}$$

2> $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i-3j+6k$

sol^n :⇒ $\phi = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

i.e $\nabla\phi = (4z^3 - 6xy^2z)i - (6x^2yz)j + (12xz^2 - 3x^2y^2)k$

$[\nabla\phi]_{(2,-1,2)} = 8i + 48j + 84k$

The unit vector in the direction of $2i-3j+6k$ is,

$$\hat{n} = \frac{2i-3j+6k}{\sqrt{4+9+36}} = \frac{2i-3j+6k}{7}$$

The required directional derivative is

$$\nabla\phi\cdot\hat{n} = (8i+48j+84k)\cdot\frac{2i-3j+6k}{7}$$

$$= \frac{16-144+504}{7} = \boxed{\frac{376}{7}}$$

3> Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at $(2,-1,2)$

sol^n :⇒ The angle between the surfaces is defined to be equal to the angle between their normals and we know that $\nabla\phi$ is a vector normal to the surface.

we have the Equation of the two surfaces given by $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$

Let $\phi_1 = x^2+y^2+z^2$ and $\phi_2 = x^2+y^2-z$

we have, $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\therefore \nabla\phi_1 = 2xi + 2yj + 2zk$ and $\nabla\phi_2 = 2xi + 2yj - k$

$[\nabla\phi_1]_{(2,-1,2)} = 4i - 2j + 4k$

$\therefore [\nabla\phi_1]_{(2,-1,2)} = 2(2i - j + 2k)$

$[\nabla\phi_2]_{(2,-1,2)} = 4i - 2j - k$

If θ is the angle between these two normals,

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$\cos\theta = \frac{2(8+2-2)}{\sqrt{36}\cdot\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

4> Find the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol^n: Let $\phi = xy - z^2$ and we know that $\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = yi + xj - 2zk$$

$\therefore [\nabla\phi]_{(4,1,2)} = i + 4j - 4k = \vec{A}$

$[\nabla\phi]_{(3,3,-3)} = 3i + 3j + 6k = 3(i + j + 2k) = \vec{B}$

If θ is the angle between $\vec{A}$ & $\vec{B}$, $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$

Now, $$\cos\theta = \frac{3(1+4-8)}{\sqrt{1+16+16}\cdot\sqrt{3^2(1+1+4)}} = \frac{-3}{\sqrt{33}\cdot\sqrt{6}} = \frac{-1}{\sqrt{22}}$$

Thus, $\cos\theta = \dfrac{-1}{\sqrt{22}}$ or $\theta = \pi \pm \cos^{-1}\left(\dfrac{1}{\sqrt{22}}\right)$

5> If $\vec{A} = 2x^2 i - 3yz i + xz^2 k$ and $\phi = 2z - x^3 y$ compute $\vec{A}\cdot\nabla\phi$ and $\vec{A}\times\nabla\phi$ at $(1,-1,1)$

Sol^n:- $\nabla\phi = \dfrac{\partial\phi}{\partial x} i + \dfrac{\partial\phi}{\partial y} j + \dfrac{\partial\phi}{\partial z} k$, $\phi = 2z - x^3 y$

$\nabla\phi = -3x^2 y\, i - x^3 j + 2k$

$[\nabla\phi]_{(1,-1,1)} = 3i - j + 2k$

Now, $[\vec{A}]_{(1,-1,1)} = 2i + 3j + k$.

$\therefore [\vec{A}\cdot\nabla\phi]_{(1,-1,1)} = (3)(2) + (-1)(3) + (2)(1) = \boxed{5}$

Also, $\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

$= i(6+1) - j(4-3) + k(-2-9)$

$= 7i - j - 11k$.

6> Find div. $\vec{F}$ and curl. $\vec{F}$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

Soln :⇒ Let, $\phi = x^3+y^3+z^3-3xyz$

$\vec{F} = \text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$\vec{F} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

$\nabla.\vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \{(3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy) \cdot k\}$

$\nabla.\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$\boxed{\nabla.\vec{F} = 6(x+y+z)}$

Also, curl.$\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$

$= i\left\{\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right\}$

$-j\left\{\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right\}$

$+k\left\{\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right\}$

$\nabla \times \vec{F} = i\{-3x-(-3x)\} - j\{-3y-(-3y)\} + k\{-3z-(-3z)\}$.

$\boxed{\nabla \times \vec{F} = 0}$

7> If $\vec{F} = \nabla(xy^3z^2)$ find div. $\vec{F}$ and curl. $\vec{F}$ at the point $(1,-1,1)$

Soln: Let $\phi = xy^3z^2$

$\therefore \vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$= y^3z^2 i + 3xy^2z^2 j + 2xy^3z k$

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\cdot(y^3z^2 i + 3xy^2z^2 j + 2xy^3z k)$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$\nabla\cdot\vec{F} = 2xy(3z^2 + y^2)$$

div $\vec{F}$ at $(1,-1,1)$

$$\boxed{\nabla\cdot\vec{F} = -8}$$

Also, curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

$$\nabla\times\vec{F} = i(6xy^2z - 6xy^2z) - j(2y^3z - 2y^3z) + k(3y^2z^2 - 3y^2z^2)$$

$$\boxed{\nabla\times\vec{F} = 0}$$

8> If $\vec{F} = (x+y+1)i + j - (x+y)k$, show that $\vec{F}\cdot \text{curl}\,\vec{F} = 0$

Sol^n :=> curl$\cdot\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -x-y \end{vmatrix}$

$$\text{Curl}\cdot\vec{F} = i(-1-0) - j(-1-0) + k(0-1)$$

i.e. $\text{curl}\cdot\vec{F} = -i + j - k$

$\therefore \vec{F}\cdot\text{curl}\,\vec{F} = \{(x+y+1)i + j - (x+y)k\}\cdot(-i+j-k)$

$$\vec{F}\cdot\text{curl}\cdot\vec{F} = (x+y+1)(-1) + (1)(1) + (x+y)(1)$$

$$\vec{F}\cdot\text{curl}\,\vec{F} = -x - y - 1 + 1 + x + y$$

$$\boxed{\vec{F}\cdot\text{curl}\,\vec{F} = 0}$$

9> If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ prove the following

i> $\nabla(r^n) = n\cdot r^{n-2}\,\vec{r}$

ii> $\nabla\cdot(r^n\cdot\vec{r}) = (n+3)\cdot r^n$

iii> $\nabla\times(r^n\cdot\vec{r}) = \vec{0}$

iv> $\nabla^2(r^n) = n(n+1)\cdot r^{n-2}$

i) $\nabla(r^n) = n.r^{n-2}\vec{r}$

Sol^n: $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Diff w.r.to x partially

$2r.\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(r^n) = \left(\Sigma.\frac{\partial}{\partial x} i\right)(r^n)$

$= \Sigma n.r^{n-1}.\frac{\partial r}{\partial x} i = \Sigma n.r^{n-1}\left(\frac{x}{r}\right) i$

$\nabla(r^n) = \Sigma n.r^{n-2} x i = n.r^{n-2}\Sigma x i$

$\boxed{\nabla(r^n) = n.r^{n-2}.\vec{r}}$

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ii) $\nabla.(r^n.\vec{r}) = (n+3).r^n$

Sol^n: $r^n.\vec{r} = r^n.\Sigma x i = \Sigma(r^n.x) i$

$\nabla(r^n.\vec{r}) = \left(\Sigma.\frac{\partial}{\partial x} i\right).\Sigma(r^n.x) i$

$= \Sigma.\frac{\partial}{\partial x}(r^n.x)$

$\nabla(r^n.\vec{r}) = \Sigma\left(r^n + n.r^{n-1}.\frac{\partial r}{\partial x}.x\right)$

$= \Sigma\left(r^n + n.r^{n-1}.\frac{x}{r}.x\right)$

$= \Sigma(r^n + n r^{n-2} x^2)$

$\nabla(r^n.\vec{r}) = (r^n + n r^{n-2} x^2) + (r^n + n r^{n-2} y^2) + (r^n + n r^{n-2} z^2)$

$\nabla(r^n.\vec{r}) = 3r^n + n r^{n-2}(x^2+y^2+z^2)$

$= 3r^n + n r^{n-2}.r^2 = \boxed{(n+3) r^n}$

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iii) $\nabla\times(r^n.\vec{r}) = \vec{0}$

$r^n.\vec{r} = r^n.\Sigma x i = \Sigma(r^n.x) i$

$$\nabla\times(r^n.\vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$= \Sigma i\left\{\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n.y)\right\}$

$$\nabla \times (r^n.\vec{r}) = \Sigma i \left\{ n.r^{n-1} \frac{\partial r}{\partial y} z - n r^{n-1}. \frac{\partial r}{\partial z} y \right\}$$

$$\nabla \times (r^n.\vec{r}) = \Sigma i \left\{ n\, r^{n-1} \frac{y}{r}. z - n.r^{n-1}. \frac{z}{r} y \right\}$$

$$\nabla \times (r^n.\vec{r}) = \Sigma i \, (n r^{n-2} yz - n r^{n-2} yz)$$

$$= \boxed{\vec{0}}$$

iv) $\nabla^2 (r^n) = n(n+1). r^{n-2}$

Sol^n: $\nabla^2(r^n) = \Sigma . \frac{\partial^2}{\partial x^2} (r^n) = \Sigma \frac{\partial}{\partial x} . \frac{\partial}{\partial x} (r^n)$

$$\nabla^2(r^n) = \Sigma . \frac{\partial}{\partial x} \left(n.r^{n-1} \frac{\partial r}{\partial x} \right) = \Sigma \frac{\partial}{\partial x} \left\{ n.r^{n-1} \left(\frac{x}{r} \right) \right\}$$

$$\nabla^2(r^n) = \Sigma \frac{\partial}{\partial x} (n r^{n-2}. x) = n \, \Sigma \left(r^{n-2} + (n-2). r^{n-3} \frac{\partial r}{\partial x} x \right)$$

$$\nabla^2(r^n) = n \, \Sigma \left(r^{n-2} + (n-2) r^{n-3}. \frac{x}{r}. x \right)$$

$$= n \, \Sigma \, (r^{n-2} + (n-2) r^{n-4} x^2)$$

$$\nabla^2(r^n) = n \{ (r^{n-2} + (n-2). r^{n-4}. x^2) + (r^{n-2} + (n-2) r^{n-4} y^2) + (r^{n-2} + (n-2) r^{n-4}. z^2) \}$$

$$\nabla^2(r^n) = n \{ 3 r^{n-2} + (n-2) r^{n-4} (x^2+y^2+z^2) \}$$

$$\nabla^2(r^n) = n \{ 3 r^{n-2} + (n-2) r^{n-4} r^2 \}$$

$$\boxed{\nabla^2(r^n) = n(n+1). r^{n-2}}$$

Solenoidal and Irrotational vectors :⇒

A vector $\vec{F}$ is said to be solenoidal if 'div. $\vec{F} = 0$' and irrotational if "curl. $\vec{F} = 0$"

Irrotational vector field is also called as. Conservative field or potential field.

when $\vec{F}$ is irrotational there always exists a Scalar point fun^n ϕ such that $\nabla \phi = \vec{F}$. Then ϕ is called a scalar potential of $\vec{F}$.

Problems :-

1) Show that $\vec{F} = (y+z)i + (z+x)j + (x+y)k$ is irrotational. Also find a scalar point fun ϕ such that $\vec{F} = \nabla\phi$.

Sol^n :- we have to show that curl. $\vec{F} = \nabla \times \vec{F} = 0$

$$\text{curl}.\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$\text{Curl}.\vec{F} = i(1-1) - j(1-1) + k(1-1) = 0.$$

Now Let us consider, $\nabla\phi = \vec{F}$

i.e $\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (y+z)i + (z+x)j + (x+y)k$

$\Rightarrow \frac{\partial\phi}{\partial x} = y+z$, $\phi = \int (y+z).dx + f_1(y,z)$

i.e $\phi = xy + xz + f_1(y,z)$ ——— ①

$\Rightarrow \frac{\partial\phi}{\partial y} = z+x$, $\phi = \int (z+x).dy + f_2(x,z)$

i.e $\phi = yz + xy + f_2(x,z)$ ——— ②

$\Rightarrow \frac{\partial\phi}{\partial z} = x+y$, $\phi = \int (x+y).dz + f_3(x,y)$

i.e $\phi = xz + yz + f_3(x,y)$ ——— ③

It is the union of ①, ② & ③ avoiding repetitions.

Let us choose, $f_1(y,z) = yz$

$f_2(x,x) = xz$

$f_3(x,y) = xy$ from ①, ② & ③.

Thus. the required,

$$\boxed{\phi = xy + yz + zx}$$

2) Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)K$ is a conservative force field. Find its scalar potential.

Soln :⇒ $\nabla \times \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy^2+yz) & (2x^2y+xz+2yz^2) & (2y^2z+xy) \end{vmatrix}$

$\nabla \times \vec{F} = i(4yz + x - x - 4yz) - j(y-y) + K(4xy + z - 4xy - z) = \vec{0}$

Thus, $\vec{F}$ is conservative.

Now we have to find ϕ such that $\nabla\phi = \vec{F}$.

we obtain, $\boxed{\phi = x^2y^2 + y^2z^2 + xyz}$

3) Find the value of the constant 'a' such that the vector field, $\vec{F} = (axy - z^3)i + (a-2)x^2j + (1-a)xz^2K$ is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$.

Soln: we have to find 'a' such that curl $\vec{F} = \vec{0}$.

i.e. $\nabla \times \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}$

i.e $i(0-0) - j\{(1-a)z^2 + 3z^2\} + K\{(a-2)2x - ax\} = \vec{0}$

$(a-4)z^2j + (a-4)xK = \vec{0}$, $\boxed{a=4}$.

The above eqn is identically satisfied when $a=4$

Now consider, $\nabla\phi = (\vec{F})_{a=4}$

i.e $\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K = (4xy - z^3)i + (2x^2)j - 3xz^2K$

we have.

$\frac{\partial\phi}{\partial x} = 4xy - z^3$ (Integrate w.r.to x).

$\phi = 2x^2y - xz^3 + f_1(y,z)$ —— ①

$\frac{\partial \phi}{\partial y} = 2x^2$, Integrating w.r.to y

$\therefore \phi = 2x^2 y + f_2(x,z)$ —— ②

$\frac{\partial \phi}{\partial z} = -3xz^2$. Integrating w.r.to z.

$\therefore \phi = -xz^3 + f_3(x,y)$ —— ③

Let, $f_1(y,z) = 0$, $f_2(x,z) = -xz^3$, $f_3(x,y) = 2x^2y$

from ①, ② & ③.

Thus, $\boxed{\phi = 2x^2y - xz^3}$

4) Find constants a and b such that $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational. Also find a Scalar function ϕ such that $\vec{F} = \nabla\phi$.

Soln :- we have to find a and b such that curl. $\vec{F} = 0$

we obtain $a = 6$, $b = 3$

$$\phi = 3x^2y + xz^3 - yz.$$

5) If $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (x + cy + 2z)k$. find a, b, c such that curl. $\vec{F} = 0$ and then find ϕ such that $\vec{F} = \nabla\phi$.

Soln :-

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix} = 0$$

i.e $i(c+1) - j(1-a) + k(b-1) = 0$

$a = 1$, $b = 1$, $c = -1$

Now consider.

$\nabla\phi = \vec{F}$, when $a = 1$, $b = 1$, $c = -1$

i.e $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (x+y+z) i + (x+2y-z) j$
$+ (x-y+2z) k$

$\Rightarrow \frac{\partial \phi}{\partial x} = x+y+z$

$\phi = \int (x+y+z)\, dx + f_1(y,z)$

i.e $\phi = \frac{x^2}{2} + xy + xz + f_1(y,z)$ —— ①

$\Rightarrow \frac{\partial \phi}{\partial y} = x+2y-z$

$\phi = \int (x+2y-z)\, dy + f_2(x,z)$

i.e $\phi = xy + y^2 - yz + f_2(x,z)$ —— ②

$\Rightarrow \frac{\partial \phi}{\partial z} = x-y+2z$.

$\phi = \int (x-y+2z)\, dz + f_3(x,y)$

i.e $\phi = xz - yz + z^2 + f_3(x,y)$ —— ③

Thus the required.

$$\boxed{\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2}$$

Vector Integration

Vector Integration: Line integrals, surface integrals, Applications to work done by a force and flux, statement of Green's and stokes thm problems.

Vector line integral :-
consider a curve C in space which consist of infinitesimally small line Elements of length dr. Then the line integral of a vector $\vec{A}(x, y, z)$ along the curve C is defined to be the sum of the scalar products of $\vec{A}$ and $d\vec{r}$ and is represented by $\int_C \vec{A} \cdot d\vec{r}$
If C is a closed curve which donot intersect any where, the line integral around C denoted by $\oint \vec{A} \cdot d\vec{r}$

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then $\oint_C \vec{F} \cdot d\vec{r}$ represents the total workdone by the force
It also represents the circulation of $\vec{F}$ about C where $\vec{F}$ represents the velocity of the fluid. $\vec{F}$ is said to be irrotational if $\oint_C \vec{F} \cdot d\vec{r} = 0$.

1) If $\vec{F} = xyi + yzj + zxk$, Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$

sol^n: we have $\vec{F} = xyi + yzj + zxk$ and $\vec{r} = xi + yj + zk$
will give $d\vec{r} = dxi + dyj + dzk$

$$\vec{F} \cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$$

Since $x = t, y = t^2, z = t^3$ by data, we obtain

$$dx = dt, \quad dy = 2t \cdot dt, \quad dz = 3t^2 \cdot dt$$

$$\vec{F} \cdot d\vec{r} = t^3\,dt + t^5(2t)\,dt + t^4(3t^2)\,dt$$

$$\vec{F} \cdot d\vec{r} = (t^3 + 2t^6 + 3t^6)\,dt$$

$$\vec{F} \cdot d\vec{r} = (t^3 + 5t^6)\,dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt = \left[\frac{t^4}{4}\right]_{-1}^{1} + 5\left[\frac{t^7}{7}\right]_{-1}^{1}$$

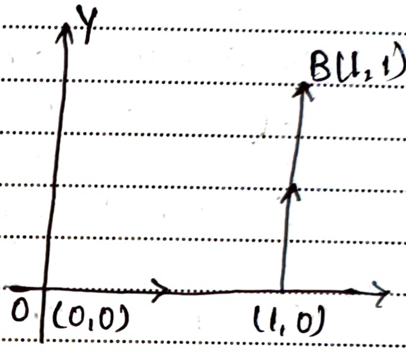
Thus $\int_C \vec{F}\cdot d\vec{r} = \left(\frac{1}{4}-\frac{1}{4}\right)+5\left(\frac{1}{7}+\frac{1}{7}\right) = \frac{10}{7}$

2> Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\hat{i} + (x^2+y^2)\hat{j}$ along

i> the path of the straight line from $(0,0)$ to $(1,0)$ and then to $(1,1)$

ii> the straight line joining the origin and $(1,2)$

Soln:- $\int_C \vec{F}\cdot d\vec{r} = \int_C xy\,dx + (x^2+y^2)\,dy$ —— ①



i> $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$ —— ②

Along OA : $y=0$

$\therefore dy = 0$ and $0 \le x \le 1$

From ① $\int_{OA} \vec{F}\cdot d\vec{r} = 0$ —— ③

Along AB : $x=1$ $\therefore dx=0$ and $0 \le y \le 1$. Again from ①

$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=0}^{1} 0 + (1+y^2)\,dy = \left[y+\frac{y^3}{3}\right]_{y=0}^{1} = 1+\frac{1}{3} = \frac{4}{3}$ —— ④

Using ③ & ④ in ② we obtain $\int_C \vec{F}\cdot d\vec{r} = 0+\frac{4}{3} = \frac{4}{3}$

ii> C is the straight line joining $(0,0)$ and $(1,2)$

The eqn of the line is given by $\frac{y-0}{x-0} = \frac{2-0}{1-0}$

i.e. $y = 2x$ $\therefore dy = 2dx$ and x varies from 0 to 1

Hence from ① $\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} x\cdot 2x\,dx + (x^2+4x^2)2\,dx$

Thus, $\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} 12x^2\,dx = 12\left[\frac{x^3}{3}\right]_0^1 = 4$

3> If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, Evaluate $\int \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve given by $x=t$, $y=t^2$, $z=t^3$

Sol^n: $\vec{F} = (3x^2+6y)i - 14yzj + 20xz^2k$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

$\vec{F}\cdot d\vec{r} = (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$

Since $x=t,\ y=t^2,\ z=t^3$ we obtain $dx=dt,\ dy=2t\,dt,$ $dz = 3t^2\cdot dt$

$\therefore \vec{F}\cdot d\vec{r} = (3t^2+6t^2)\,dt - (14t^5)\,2t\,dt + (20t^7)\,3t^2\,dt$

i.e $\vec{F}\cdot d\vec{r} = (9t^2 - 28t^6 + 60t^9)\,dt$; $0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$

Thus, $$\int_C \vec{F}\cdot d\vec{r} = \left[9\cdot\frac{t^3}{3} - 28\cdot\frac{t^7}{7} + 60\,\frac{t^{10}}{10}\right]_{t=0}^{1} = 3-4+6 = 5$$

4> If $\vec{F} = x^2 i + xy\,j$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along

i> the line $y=x$ ii> the parabola $y=\sqrt{x}$

Sol^n: $\vec{F}\cdot d\vec{r} = x^2\,dx + xy\,dy$

i> Along $y=x$: we have $0\le x\le 1$ and $dy=dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} x^2\,dx + \int_{x=0}^{1} x^2\,dx$$

$$= \int_0^1 2x^2\,dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

ii> Along $y=\sqrt{x}$, $y^2 = x$ and $2y\,dy = dx$, $0\le y\le 1$

Thus, $$\int_C \vec{F}\cdot d\vec{r} = \int_{y=0}^{1} 2y^5\,dy + \int_{y=0}^{1} y^3\,dy$$

$$= \left[\frac{y^6}{3}\right]_0^1 + \left[\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

5> Find the total work done by the force represented by $\vec{F} = 3xy\,i - y\,j + 2zx\,k$ in moving a particle round the circle $x^2+y^2=4$

Sol^n: Total work done $W = \int_C \vec{F}\cdot d\vec{r}$

$x^2+y^2=4$ can be represented in the parametric form
$x=2\cos\theta$, $y=2\sin\theta$ and $z=0$, $0\le\theta\le 2\pi$

$$W=\int_C \vec{F}\cdot d\vec{r}=\int 3xy\,dx-y\,dy+2zx\,dz$$

$$W=\int_{\theta=0}^{2\pi} 3(4\cos\theta\cdot\sin\theta)(-2\sin\theta)\,d\theta-\int_{\theta=0}^{2\pi}4\sin\theta\cdot\cos\theta\cdot d\theta$$

$$=-24\int_0^{2\pi}\sin^2\theta\cdot\cos\theta\cdot d\theta-2\int_0^{2\pi}\sin 2\theta\cdot d\theta$$

$\cos\theta+\cos^2\theta=1$
$\cos\theta=1-\cos^2\theta$
$\cos\theta=\sin^2\theta$
$\sin^2\theta=\cos\theta$

$$=-24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi}-2\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}=0$$

Thus the total work done is 0.

Surface integrals :→

An integral evaluated over a surface is called a surface integral. consider a surface S and a point P on it. Let $\vec{A}$ be a vector funⁿ of x, y, z defined and continuous over S.

If $\hat{n}$ is the unit outward normal to the surface S at P then the integral of the normal component of $\vec{A}$ at P (i.e $\vec{A}\cdot\hat{n}$) over the surface S is called the surface integral written as $\iint_S \vec{A}\cdot\hat{n}\,ds$ where ds is the small elemental area. To evaluate a surface integral we have to find the double integral over the orthogonal projection of the surface on one of the coordinate planes. Suppose R is the orthogonal projection of S on the xoy plane and $\hat{n}$ is the unit outward normal to S then it should be noted that $\hat{n}\cdot\hat{k}\,ds$. ($\hat{k}$ being the unit vector along z-axis) is the projection of the vectorial area element $\hat{n}\cdot ds$ on the xoy plane and this projection is equal to $dx\,dy$ which being the area element in the xoy plane
i.e to say that $\hat{A}\cdot\hat{k}\,ds=dx\cdot dy$. similarly we can argue to state that $\hat{n}\cdot\hat{j}\,ds=dz\cdot dx$ & $\hat{n}\cdot\hat{i}\,ds=dy\,dz$.
All these results hold good if we write

$\hat{n} \cdot ds = dy\, dz\, i + dz \cdot dx\, j + dx\, dy\, k$

sometimes we also write $ds = \hat{n} \cdot ds = \Sigma\, dy\, dz\, i$.

Integral theorems :→

* Green's theorem in a plane :→

Statement :→ If R is a closed region of the x-y plane bounded by a simple closed curve C and if M & N are two continuous fun^ns of x, y having continuous first order partial derivatives in the region R then

$$\int_C M \cdot dx + N \cdot dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \cdot dy$$

* Stoke's theorem :→

Statement :→ If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector fun^n then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \cdot \vec{F} \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \cdot ds$$

1> Verify Green's thm in a plane for $\int (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the region enclosed by $y = \sqrt{x}$ & $y = x^2$

Sol^n :- we shall find the points of intersection of the parabolas $y = \sqrt{x}$ and $y = x^2$

i.e $\sqrt{x} = x^2 \Rightarrow x = x^4$ or $x(x^3 - 1) = 0$

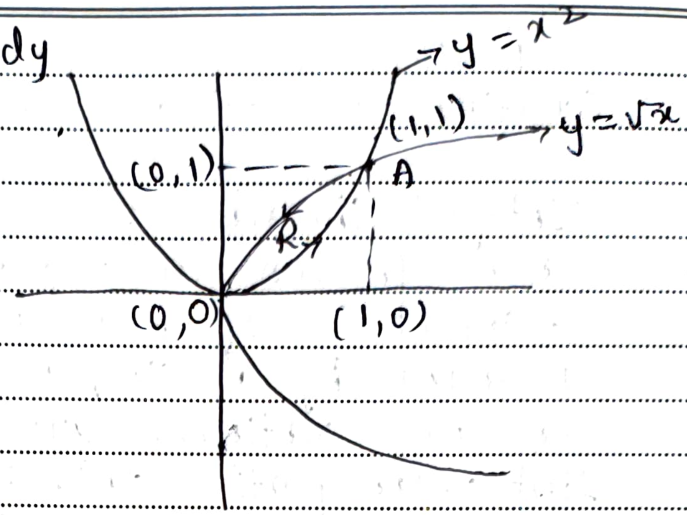
$x = 0, 1$ and hence $y = 0, 1$. The points of intersection are $(0,0)$ and $(1,1)$

Let $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

We have green's thm in plane

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$$LHS = \int_C M\,dx + N\,dy$$

$$= \int_{OA} M \cdot dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= I_1 + I_2$$

along OA $= y = x^2 \quad dy = 2x\,dx, \; 0 \leq x \leq 1$

$$I_1 = \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x)\,dx$$

$$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 = -1$$

Along AO $= y = \sqrt{x}$ or $x = y^2 \Rightarrow dx = 2y\,dy$ y varies from 1, 0

$$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy$$

$$= \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0$$

$$= 0 - \left(2 - \frac{11}{2} + 1\right) = \frac{5}{2}$$

Hence $LHS = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$

Also, $RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx$$

$$RHS = \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\, dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

$LHS = 3/2 = RHS.$

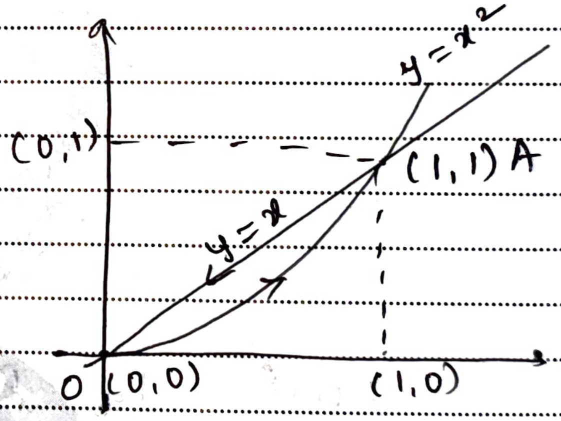
2) Verify Green's thm for $\int_C (xy + y^2)\, dx + x^2 dy$ where C is closed curve of the region bounded by $y = x$ & $y = x^2$

sol.^{n}: we shall find the points of intersection of $y = x$ and $y = x^2$

equating the RHS we have, $x = x^2$ or $x(1-x) = 0$

$x = 0, 1$

$\therefore$ $y = 0, 1$ and hence $(0,0), (1,1)$ are the points of intersection.



we have Green's thm in a plane

$$\int_C M.dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\int_C (xy + y^2)\, dx + x^2 dy$$

$$= \int_{OA} (xy + y^2)\, dx + x^2 dy + \int_{AO} (xy + y^2)\, dx + x^2 dy$$

Along OA we have $y = x^2$

$dy = 2x\, dx$ and $0 \leq x \leq 1$

$$I_1 = \int_{x=0}^{1} (x \cdot x^2 + x^4)\, dx + x^2 \cdot 2x\, dx$$

$$= \int_{x=0}^{1} (3x^3 + x^4)\, dx = 3\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Next, along AO we have $y = x$

$dy = dx \quad , \quad 1 \le x \le 0$

$$I_2 = \int_{x=1}^{0} (x \cdot x + x^2)\, dx + x^2\, dx$$

$$= \int_{x=1}^{0} 3x^2 \cdot dx = [x^3]_0^1 = -1$$

Hence $LHS = I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$

To evaluate the RHS we have

$M = xy + y^2, \quad N = x^2$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (x + 2y) = x - 2y$$

R is the region bounded by $y = x^2$ & $y = x$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} [xy - y^2]_{y=x^2}^{x}\, dx$$

$$= \int_{x=0}^{1} [(x^2 - x^2) - (x^3 - x^4)]\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

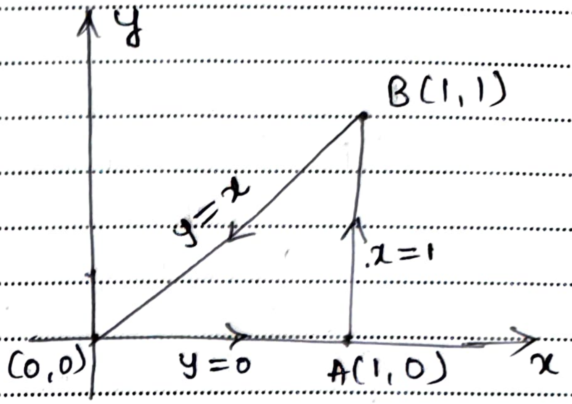
$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

$LHS = \frac{-1}{20} = RHS$

3) Evaluate $\int_C (xy - x^2)\,dx + x^2 y\,dy$ where C is the closed curve formed by $y=0$, $x=1$ and $y=x$

a) directly as a line integral

b) by Green's thm.



sol: a) Let $M = xy - x^2$, $N = x^2 y$.

$$\int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BO} M\,dx + N\,dy$$

i) Along OA : $y = 0 \Rightarrow dy = 0$ and $0 \le x \le 1$

ii) Along AB : $x = 1 \Rightarrow dx = 0$ and $0 \le y \le 1$

iii) Along BO : $y = x \Rightarrow dy = dx$ and $1 \le x \le 0$

$$\therefore \int_C M\,dx + N\,dy = \int_{x=0}^{1} -x^2\,dx + \int_{y=0}^{1} y\,dy + \int_{x=1}^{0} x^3\,dx$$

$$= -\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{-1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{-1}{12}$$

Thus $\int_C (xy - x^2)\,dx + x^2 y\,dy = \frac{-1}{12}$

b) we have green's thm $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$RHS = \iint_R (2xy + x)\,dx \cdot dy$$

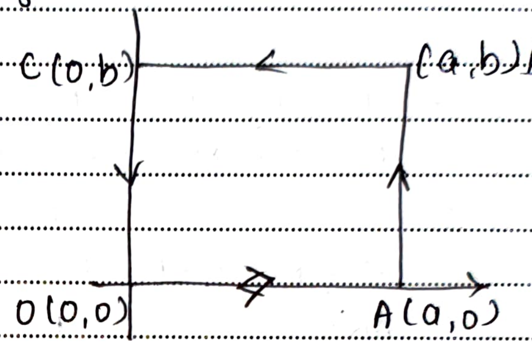
$$= \int_{x=0}^{1} \int_{y=0}^{x} (2xy - x)\,dy\,dx \quad \text{from the figure}$$

$$= \int_{x=0}^{1} \left[x y^2 - xy\right]_{y=0}^{x} dx = \int_{x=0}^{1} (x^3 - x^2)\,dx$$

Thus RHS $= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}$

1> Verify stokes thm for vector $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by $x=0$, $x=a$, $y=0$, $y=b$

sol.?: we have stoke's thm $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\cdot ds$



$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

$$\int \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

$$\int_C \vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3 + I_4$$

i> Along OA : $y=0 \Rightarrow dy=0$ and $0 \le x \le a$

$$I_1 = \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

ii> Along AB ; $x=a \Rightarrow dx=0$ and $0\le y\le b$

$$I_2 = \int_{y=0}^{b} -2ay\,dy = [-ay^2]_0^b = -ab^2$$

iii> Along BC : $y=b \Rightarrow dy=0$ and $a\le x\le 0$

$$\therefore I_3 = \int_{x=a}^{0} (x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_a^0 = \frac{-a^3}{3} - ab^2$$

iv> Along CO : $x=0 \Rightarrow dx=0$ and $b\le y\le 0$

$$I_4 = \int_{y=b}^{0} 0\cdot dy = 0$$

Hence $\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2$

Now, $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = k(-2y-2y) = -4yk$

$(\nabla\times\vec{F})\,\hat{n}\,.ds = (-4yk)\cdot(dy\,dz\,\hat{i} + dz\,dx\,j + dx\,dy\,k)$

$= -4y\cdot dx\,dy$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,.ds = \iint_R -4y\,dx\,dy$$

$$= -4 \int_{x=0}^{a} \int_{y=0}^{b} y \, dy \, dx$$

$$= -4 \int_{x=0}^{a} \left[\frac{y^2}{2} \right]_0^b dx = -2b^2 [x]_0^a$$

$$= -2ab^2$$

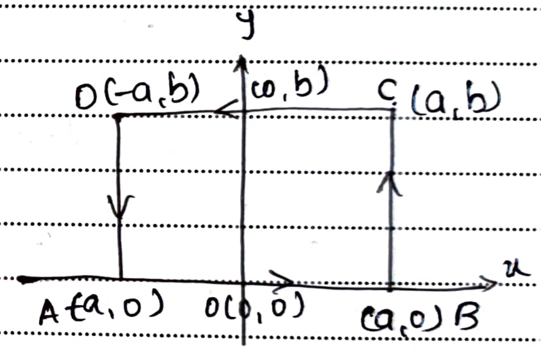
Hence we have $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = -2ab^2$

2> Verify stokes thm for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0$ and $y=b$

sol^n: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \, \vec{F} \cdot \hat{n} \, ds$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r}$$

$$= I_1 + I_2 + I_3 + I_4$$



$\vec{F} \cdot d\vec{r} = (x^2+y^2) \, dx - 2xy \, dy$

i> Along AB : $y=0 \Rightarrow dy=0$: $-a \le x \le a$

$$I_1 = \int_{-a}^{a} x^2 dx = \left[\frac{x^3}{3} \right]_{-a}^{a}$$

$$= \frac{a^3}{3} - \left(\frac{-a^3}{3} \right) = \frac{2a^3}{3}$$

ii> Along BC : $x=a$ $dx=0$ $0 \le y \le b$

$$I_2 = \int_{y=0}^{b} -2ay \, dy = [-ay^2]_0^b = -ab^2$$

iii> Along CD : $y=b \Rightarrow dy=0$: $a \le x \le -a$

$$I_3 = \int_{x=a}^{-a} [x^2+b^2] \, dx = \left[\frac{x^3}{3} + b^2 x \right]_{x=a}^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2 \right) - \left(\frac{a^3}{3} + ab^2 \right) = -\frac{2a^3}{3} - 2ab^2$$

iv> Along DA : $x=-a \Rightarrow dx=0$ $b \le y \le 0$

$$I_4 = \int_{y=b}^{0} 2ay \, dy = [ay^2]_b^0 = -ab^2$$

Hence $\int_C \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = \underline{\underline{-4ab^2}}$

Now curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & (-2xy) & 0 \end{vmatrix} = -4yk$

$\vec{ds} = \hat{n}.ds = dy\,dz\,i + dz.dx\,j + dx.dy\,k$

Curl . $\vec{F} \cdot \hat{n}.ds = -4y.dx.dy$

$$\iint_S \text{curl}.\vec{F}.\hat{n}.ds = -4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= -4\int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx$$

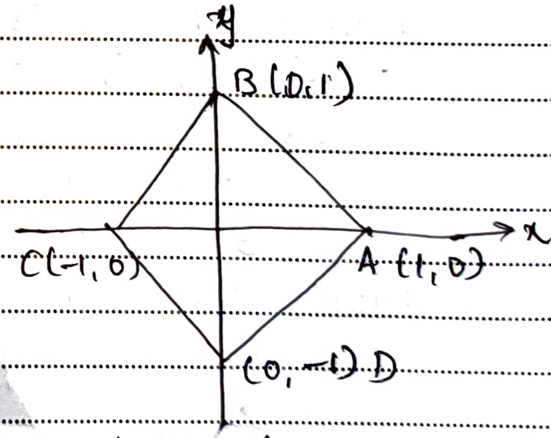
$$= -2\int_{x=-a}^{a} b^2\,dx = -2b^2[x]_{-a}^{a}$$

$$= -2b^2(2a) = \underline{-4ab^2}$$

3> Evaluate $\int_C xy\,dx + xy^2\,dy$ by stokes thm where C is the square in the x-y plane with verticles (1, 0) (-1, 0) (0, 1) (0, -1)

Sol:- we have stokes thm

$$\int_C \vec{F}.\vec{dr} = \iint_S \text{curl}.\vec{F}.\hat{n}.ds$$



From the given integral it is Evident that $\vec{F} = xy\,i + xy^2\,j$

since $\vec{dr} = dx\,i + dy\,j + dz\,k$

Hence $\int_C xy\,dx + xy^2\,dy = \int_C \vec{F}.\vec{dr}$

which is to be Evaluated by applying stoke thm

Now curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$

i.e curl $\vec{F} = (y^2 - x)k$. on Expanding the determinant

Further $\vec{ds} = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$

$$\iint_S \text{curl}\,\vec{F}.\vec{ds} = \iint (y^2-x)\,dx.dy$$

It can be clearly seen from the figure that

$-1 \le x \le 1$, & $-1 \le y \le 1$

Now $\iint_S \text{curl}.\vec{F}.d\vec{s} = \int_{x=-1}^{1}\int_{y=-1}^{1} (y^2-x)\,dy\,dx$

$\iint_S \text{curl}\,\vec{F}.ds = \int_{x=-1}^{1}\left[\frac{y^3}{3}-xy\right]_{y=-1}^{1} dx$

$\iint_S \text{curl}.\vec{F}.d\vec{s} = \int_{x=-1}^{1}\left[\left(\frac{1}{3}+\frac{1}{3}\right)-x(1+1)\right]dx$

$= \int_{x=-1}^{1}\left[\frac{2}{3}-2x\right]dx = \left[\frac{2}{3}x-x^2\right]_{-1}^{1}$

$= \frac{2}{3}(1+1)-(1-1) = \frac{4}{3}$

Thus $\int_C xy\,dx + xy^2\,dy = \frac{4}{3}$

Problems on workdone by the force.

~~1) S.T $\vec{F} = (2xy+z^3)i + x^2j + 3z^2xk$ is~~ irr

1) Find the workdone in moving a particle in the force field $F = 3x^2 i + (2xz-y)j + zk$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol:- $\vec{F} = 3x^2 i + (2xz-y)j + zk$

$\int \vec{F}.d\vec{r} = \int (3x^2)dx + (2xz-y)dy + z\,dz$

Straight line from $(0,0,0)$ to $(2,1,3)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$x = 2t \qquad y = t \qquad z = 3t$

$dx = 2dt \qquad dy = dt \qquad dz = 3dt$

$\int \vec{F}.d\vec{r} = \int 3x^2\,dx + (2xz-y)dy + z\,dz$

$\int \vec{F}.d\vec{r} = \int 3(4t^2).2dt + (2(2t)(3t)-t)dt + 3t.3dt$

$$= \int_0^1 24\,t^2\,dt + (12t^2 - t)\,dt + 9t\,dt$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt$$

$$= \int_0^1 24\,t^2 + 12t^2 + 8t\,dt \qquad (0,1) \rightarrow \text{minimum}$$

$$= 24\left[\frac{t^3}{3}\right] + 12\left[\frac{t^3}{3}\right] + 8\left[\frac{t^2}{2}\right]$$

$$= 24\left[\frac{1}{3}\right] + 12\left[\frac{1}{3}\right] + 8\left[\frac{1}{2}\right] = 16 \,//$$

2) Find the workdone by $\vec{r} = (2x - y - z)\,\hat{i} + (x + y - z)\,\hat{j} + (3x - 2y - 5z)\,\hat{k}$ along a curve c in xy plane given by $x^2 + y^2 = 9$, $z = 0$

Sol:- $\vec{F}\cdot d\vec{r} = [2x - y - z]\,dx + (x + y - z)\,dy + (3x - 2y - 5z)\,dz$

$x - y$ plane, $z = 0$, $dz = 0$

$\vec{F}\cdot d\vec{r} = (2x - y)\,dx + (x + y)\,dy$

$x^2 + y^2 = 9 \Rightarrow x^2 + y^2 = r^2$ where $x = r\cos\theta$

$y = r\sin\theta$

$\theta \rightarrow 0$ to 2π

$\downarrow$

$x = 3\cos\theta \qquad y = 3\sin\theta$

$dx = -3\sin\theta\,d\theta \qquad dy = 3\cos\theta\,d\theta$

$$\int F\cdot dr = \int_0^{2\pi} (2x - y)\,dx + (x + y)\,dy$$

$$= \int_0^{2\pi} [2(3\cos\theta) - 3\sin\theta](-3\sin\theta\,d\theta) + (3\cos\theta + 3\sin\theta)(3\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} [6\cos\theta - 3\sin\theta](-3\sin\theta\,d\theta) + (9\cos^2\theta\,d\theta + 9\sin\theta\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} [-18\sin\theta\cdot\cos\theta + 9\sin^2\theta)\,d\theta + (9\cos^2\theta\,d\theta + 9\sin\theta\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} [-18\sin\theta\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cdot\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} -9\sin\theta\cos\theta + 9(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} (-9 \sin\theta \cdot \cos\theta + 9)\, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{9}{2} \times 2 \sin\theta \cdot \cos\theta + 9\right) d\theta$$

$$= \int_0^{2\pi} \left[-\frac{9}{2} (\sin 2\theta) + 9\right] d\theta$$

$$= \frac{-9}{2} \int_0^{2\pi} \sin 2\theta + 9 \int_0^{2\pi} d\theta$$

$$= \frac{-9}{2} \left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} + 9 [\theta]_0^{2\pi}$$

$$= \frac{-9}{4} \left[-\cos 4\pi + \cos 0\right] + 9[2\pi - 0]$$

$$= \frac{-9}{4} [-1 + 1] + 18\pi$$

$$= 18\pi$$ //