Calculus

Syllabus –

Introduction to polar co-ordinates and curvature relating to EC and EE Englin Engineering applications polar co-ordinates, polar curves, angle between the radius vector and tangent, angle between two curves, pedal Equations, curvature and Radius of curvature – Cartesian, parametric, polar and pedal forms problems

self study – Center and circle of curvature, evolutes and involutes

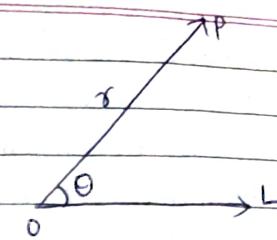
Applications : communication signals, Manufacturing of microphones and image processing.

polar co-ordinates and curvature :⇒

polar ~~co-o~~ curves. :→

We are conversant in representing the position of a point $p(x, y)$ in the Cartesian System and accordingly (x, y) are called Cartesian co-ordinates. In this topic we discuss another important System to represent a point in a plane known as the polar system.

Polar co-ordinates :⇒



Initial reference is chosen by Spotting a point O in the plane called as the pole.

A line OL drawn through O is called the initial line. If P is any given point in the plane, join the points O and P with the result an angle is formed at O.

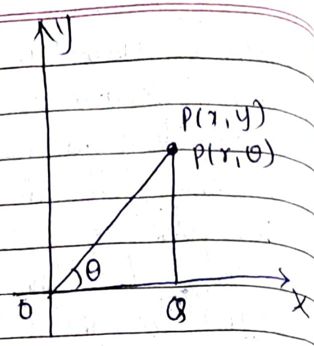
The length of OP denoted by r is called the radius vector of the point P and angle LOP denoted by θ measured in the anticlockwise direction is called the vectorial angle

The pair r and θ represented by $P = (r, \theta)$ or $P(r, \theta)$ are called as the polar coordinates of the point P

It is evident that r is positive and θ lies between 0 and 2π according as the position of the point p in the four quadrants.

We now proceed to establish the relationship between the Cartesian coordinates (x, y) and the polar coordinates (r, θ)

Let (x, y) and (r, θ) respectively represent the Cartesian and polar coordinates of any point P in the plane where the origin O is taken as the pole and the x-axis is taken as the initial line



From the figure we have $OQ = x$, $PQ = y$
Also from the right angled triangle OQP, we have

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \quad \therefore x = r\cos\theta \quad \text{—— (1)}$$

$$\sin\theta = \frac{QP}{OP} = \frac{y}{r} \quad \therefore y = r\sin\theta \quad \text{—— (2)}$$

Further by squaring and adding (1) & (2) we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 \cdot 1 = r^2$$

$$r = \sqrt{x^2+y^2} \quad \text{—— (3)}$$

Also dividing (2) by (1) we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \quad \text{or} \quad \tan\theta = \frac{y}{x}$$

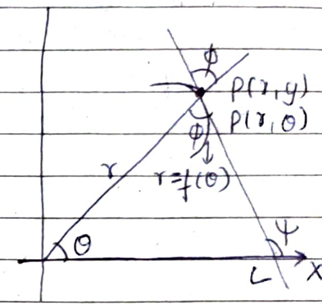
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{—— (4)}$$

The relations (1) and (2) determine the Cartesian coordinates interms of polar coordinates, where as relations (3) and (4) determine the polar coordinates interms of Cartesian coordinates.

It is evident that r is a function of θ (r depends on θ) and the equation in the form $r = f(\theta)$ or $f(r, \theta) = c$, c being a constant, is called the equation of the curve in the polar form or simply a polar curve.

Angle between radius vector and tangent

Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$
$\therefore X\hat{O}P = \theta$ and $OP = r$
Let PL be the tangent to the curve at P subtending an angle ψ with the positive direction of the initial line (x-axis) and ϕ be the angle between the radius vector OP and the tangent PL. That is $O\hat{P}L = \phi$



From the figure we have.

$$\psi = \phi + \theta.$$

(Recall from geometry that, an exterior angle is equal to the sum of the interior opposite angles)

$$\Rightarrow \tan\psi = \tan(\phi + \theta)$$

$$\text{or} \quad \tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \quad \text{—— (1)}$$

Let (x, y) be the Cartesian co-ordinates of P so that we have

$x = r\cos\theta$; $y = r\sin\theta$

Since r is a function of θ, we can as well regard these as parametric equations in terms of θ.

We also know from the Geometrical meaning of the derivative that

$\tan\psi = \frac{dy}{dx}$ = slope of the tangent PL

i.e $\tan\psi = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$ Since x & y are functions of θ.

i.e $\tan\psi = \dfrac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \dfrac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$

where $r' = \frac{dr}{d\theta}$.

Dividing both the numerator and denominator by $r'\cos\theta$ we have

$$\tan\psi = \dfrac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{-r\sin\theta}{r'\cos\theta} + \frac{r'\cos\theta}{r'\cos\theta}}$$

i.e $$\tan\psi = \dfrac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \quad \text{---} \; (2)$$

Comparing equations (1) and (2) we have

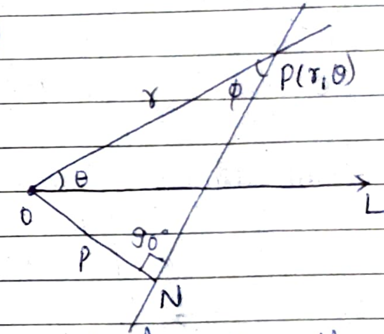
$\tan\phi = \frac{r}{r'} = \dfrac{r}{\left(\frac{dr}{d\theta}\right)}$ or $\boxed{\tan\phi = r\left(\frac{d\theta}{dr}\right)}$

Equivalently we can write in the form

$\frac{1}{\tan\phi} = \frac{1}{r}\left(\frac{dr}{d\theta}\right)$ or $\boxed{\cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)}$

Length of the perpendicular from the pole to the tangent :-

Let O be the pole and OL be the initial line. Let $P(r,\theta)$ be any point on the curve and hence we have $OP = r$ and $L\hat{O}P = \theta$



Draw $ON = p$ (say) perpendicular from the pole onto the tangent at P and let ϕ be the angle made by the radius vector with the tangent.

From the figure $O\hat{N}P = 90°$ and $L\hat{O}P = \theta$

Now from the right angled triangle ONP,

$\sin\phi = \frac{ON}{OP}$

ie $\sin\phi = \frac{p}{r}$ or $\boxed{p = r\sin\phi}$

(This Expression is the basic Expression for the length of the perpendicular P. We proceed to present the Expression for p in terms of θ in two standard forms)

we have

$$p = r\sin\phi \quad \text{——— (1)}$$

and $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$ ——— (2)

Squaring Eqⁿ (1) and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{1}{\sin^2\phi} \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

i.e $\dfrac{1}{p^2} = \dfrac{1}{r^2}(1+\cot^2\phi)$

Now using (2) we get

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

or $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ——— (3)

Further,

Let $\dfrac{1}{r} = u$

Differentiating w.r.to θ we get

$$-\frac{1}{r^2}\left(\frac{dr}{d\theta}\right) = \frac{du}{d\theta} \Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

(by Squaring)

Thus (3) now becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{——— (4)}$$

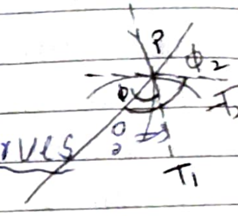
Note - The usual format of the question is as follows

i) prove with usual notations

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

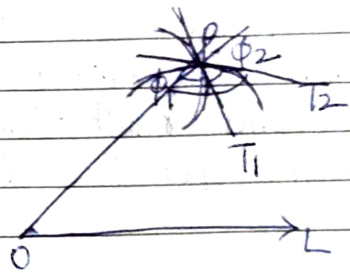
ii) prove that for the curve $r = f(\theta)$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{where } u = \frac{1}{r}$$



Angle between two polar curves

Basically we know that the angle of intersection of any two curves is equal to the angle between the tangents drawn at the point of intersection of the two curves



Let $r = f_1(\theta)$ and $r = f_2(\theta)$ be two curves intersecting at the point P

Let PT_1 and PT_2 be the tangents drawn to the curves at the point P.

It can be seen from the figure that ϕ_1 is the angle between the radius vector OP and the tangent PT_1 and ϕ_2 is the angle

made by the radius vector OP with PT_2
It can be clearly seen that the angle between the two tangents is equal to $\phi_2 - \phi_1$

$\therefore$ The acute angle of the intersection of the curves is equal to $|\phi_2 - \phi_1|$

If $|\phi_2 - \phi_1| = \frac{\pi}{2}$ then we say that the two curves intersect orthogonally.

Further if $\phi_2 - \phi_1 = \frac{\pi}{2}$ then $\phi_2 = \frac{\pi}{2} + \phi_1$

$$\therefore \tan\phi_2 = \tan\left(\frac{\pi}{2} + \phi_1\right) = -\cot\phi_1 = \frac{-1}{\tan\phi_1}$$

or $\tan\phi_1 \cdot \tan\phi_2 = -1$

This result serves as an alternative condition for the orthogonality of two polar curves.

Note :- 1) $\sin(\pi/2 - \theta) = \cos\theta$, $\cos(\pi/2 - \theta) = \sin\theta$
$\tan(\pi/2 - \theta) = \cot\theta$, $\cot(\pi/2 - \theta) = \tan\theta$

2) $\sin(\pi/2 + \theta) = \cos\theta$, $\cos(\pi/2 + \theta) = -\sin\theta$
$\tan(\pi/2 + \theta) = -\cot\theta$, $\cot(\pi/2 + \theta) = -\tan\theta$

3) $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$, $\cot\left(\frac{\pi}{4} + \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

$1 + \cos\theta = 2\cos^2(\theta/2)$ $\quad 1 - \cos\theta = 2\sin^2(\theta/2)$
$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ $\quad \cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$

Problems :-

Find the angle between the radius vector and the tangent for the following polar curves

1) $r = a(1 - \cos\theta)$ $\quad$ 2) $r^2 \cos 2\theta = a^2$
3) $r^m = a^m(\cos m\theta + \sin m\theta)$ $\quad$ 4) $\frac{1}{r} = 1 + e\cos\theta$

Solutions :- 1) $r = a(1 - \cos\theta)$

Taking log on b.s.
$\log r = \log a + \log(1 - \cos\theta)$
Diff w.r. to θ we get
$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1 - \cos\theta}$$
i.e $\cot\phi = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$

$\therefore \cot\phi = \cot\frac{\theta}{2}$ $\quad$ Thus $\phi = \frac{\theta}{2}$

2) $r^2 \cos 2\theta = a^2$
Taking log on b.s we have
$2\log r + \log(\cos 2\theta) = 2\log a$
Diff w.r. to θ we get
$$\frac{2}{r}\frac{dr}{d\theta} + \frac{(-2\sin 2\theta)}{\cos 2\theta} = 0$$
$$\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$
or $\cot\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$

Thus $\phi = \frac{\pi}{2} - 2\theta$

3) $r^m = a^m(\cos m\theta + \sin m\theta)$

Taking log on b.s

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Diff w.r. to θ we get

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{(\cos m\theta + \sin m\theta)}$$

i.e $$\frac{1}{r}\frac{dr}{d\theta} = \frac{(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

or $\cot\phi = \cot\left(\frac{\pi}{4} + m\theta\right)$

Thus, $\phi = \frac{\pi}{4} + m\theta$

4) $\frac{l}{r} = 1 + e\cos\theta$

Taking log on b.s we have

$\log l - \log r = \log(1 + e\cos\theta)$

Diff w.r. to we get

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta}$$

i.e $\cot\phi = \frac{e\sin\theta}{1 + e\cos\theta}$

or $\tan\phi = \frac{1 + e\cos\theta}{e\sin\theta}$ (This cannot be simplified)

Thus, $\phi = \tan^{-1}\left(\frac{1 + e\cos\theta}{e\sin\theta}\right)$

Find the angle between the radius vector and the tangent as indicated

5) $r = a(1 + \cos\theta)$ at $\theta = \pi/3$

6) $r\cos^2(\theta/2) = a$ at $\theta = 2\pi/3$

7) $2a/r = 1 - \cos\theta$ at $\theta = 2\pi/3$

8) $r = a(1 + \sin\theta)$ at $\theta = \pi/2$

Solutions :→ 5) $r = a(1 + \cos\theta)$

$\log r = \log a + \log(1 + \cos\theta)$

Diff w.r. to θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1 + \cos\theta}$$

$$= \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

i.e $\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$

At $\theta = \frac{\pi}{3}$, $\phi = \frac{\pi}{2} + \frac{\pi}{6}$ or $\phi = \frac{2\pi}{3} = 120°$

Thus $\phi = \frac{2\pi}{3}$

6) $r\cos^2(\theta/2) = a$

$\Rightarrow \log r + 2\log\cos(\theta/2) = \log a$

Diff w.r. to θ we get

$$\frac{1}{r}\frac{dr}{d\theta} + 2\,\frac{-1/2\cdot\sin(\theta/2)}{\cos(\theta/2)} = 0$$

or $\frac{1}{r}\frac{dr}{d\theta} = \tan(\theta/2)$

i.e $\cot\phi = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$\Rightarrow \phi = \frac{\pi}{2} - \frac{\theta}{2}$

At $\theta = \frac{2\pi}{3}$, $\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} = 30^\circ$

Thus, $\phi = \frac{\pi}{6}$.

7) $\frac{2a}{r} = 1 - \cos\theta$

$\Rightarrow \log 2a - \log r = \log(1-\cos\theta)$

Diff w.r.to θ we get

$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$

i.e $-\cot\phi = \cot(\theta/2)$ or $\cot(-\phi) = \cot(\theta/2)$

$\Rightarrow \phi = -\theta/2$

At $\theta = \frac{2\pi}{3}$, $\phi = -\frac{\pi}{3} = -60^\circ$

Thus $\phi = -\frac{\pi}{3}$

8) $r = a(1+\sin\theta)$

$\Rightarrow \log r = \log a + \log(1+\sin\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$ or $\cot\phi = \frac{\cos\theta}{1+\sin\theta}$

$\theta = \frac{\pi}{2}$, $\cot\phi = \frac{0}{1+1} = 0$ or $\cot\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$

Thus $\phi = \frac{\pi}{2}$

Note :- we can simplify RHS and explicitly obtain ϕ as follows

$\cot\phi = \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2) + \sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}$

$\cot\phi = \frac{[\cos(\theta/2) - \sin(\theta/2)][\cos(\theta/2) + \sin(\theta/2)]}{[\cos(\theta/2) + \sin(\theta/2)]^2}$

$= \frac{[\cos(\theta/2) - \sin(\theta/2)]}{[\cos(\theta/2) + \sin(\theta/2)]}$

$= \frac{\cos(\theta/2)[1 - \tan(\theta/2)]}{\cos(\theta/2)[1 + \tan(\theta/2)]}$

i.e $\cot\phi = \frac{1 - \tan(\theta/2)}{1 + \tan(\theta/2)}$

or $\cot\phi = \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\Rightarrow \phi = \frac{\pi}{4} + \frac{\theta}{2}$

If we put $\theta = \frac{\pi}{2}$, we obtain $\phi = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

Show that the following pairs of curves intersect each other orthogonally

9) $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$

10) $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$

11) $r^n = a^n(\cos n\theta)$ and $r^n = b^n \sin n\theta$

12> $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$

13> $r = 4\sec^2(\theta/2)$ and $r = 9\,\mathrm{cosec}^2(\theta/2)$

14> $r = ae^{\theta}$ and $r \cdot e^{\theta} = b$

solutions :→ 9>

$r = a(1+\cos\theta)$: $r = b(1-\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$: $\log r = \log b + \log(1-\cos\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}$: $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

i.e $\cot\phi_1 = -\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$: $\cot\phi_2 = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$

i.e $\cot\phi_1 = -\tan(\theta/2)$: $\cot\phi_2 = \cot(\theta/2)$

$= \cot(\pi/2 + \theta/2)$ $\qquad \phi_2 = \theta/2$

$\phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$

∴ angle of intersection $= |\phi_1 - \phi_2|$

$= |\pi/2 + \theta/2 - \theta/2| = \pi/2$.

Thus the curves intersect each other orthogonally

Similar problem :→

Find the angle between the curves $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$

⇒ proceeding on the same line we obtain

$\phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$ and $\phi_2 = \frac{-\theta}{2}$

... betn the curves $|\phi_1 - \phi_2| = \pi$

10> $r = a(1+\sin\theta)$, $r = a(1-\sin\theta)$

⇒ $\log r = \log a + \log(1+\sin\theta)$: $\log r = \log a + \log(1-\sin\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$; $\frac{1}{r}\frac{dr}{d\theta} = \frac{-\cos\theta}{1-\sin\theta}$

i.e $\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$; $\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$

we have

$\tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$ and $\tan\phi_2 = \frac{1-\sin\theta}{-\cos\theta}$

∴ $\tan\phi_1 \cdot \tan\phi_2 = \frac{1-\sin^2\theta}{-\cos^2\theta} = \frac{\cos^2\theta}{-\cos^2\theta} = -1$

Thus the curves intersect each other orthogonally

11> $r^n = a^n \cos n\theta$: $r^n = b^n \sin n\theta$

⇒ $n\log r = n\log a + \log(\cos n\theta)$: $n\log r = n\log b + \log(\sin n\theta)$

Diff these w.r.to θ we get

$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta}$. $\frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}$

i.e $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$: $\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$

i.e $\cot\phi_1 = \cot(\frac{\pi}{2} + n\theta)$: $\cot\phi_2 = \cot n\theta$

⇒ $\phi_1 = \pi/2 + n\theta$: $\phi_2 = n\theta$

∴ $|\phi_1 - \phi_2| = \frac{\pi}{2} + n\theta - n\theta = \frac{\pi}{2}$

Thus the curves intersect each other orthogonally

12) $r^2 \sin 2\theta = a^2$; $r^2 \cos 2\theta = b^2$

$\Rightarrow 2\log r + \log(\sin 2\theta) = 2\log a$; $2\log r + \log(\cos 2\theta) = 2\log b$

Diff these w.r.to θ we get

$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$; $\frac{2}{r}\frac{dr}{d\theta} - \frac{2\sin 2\theta}{\cos 2\theta} = 0$

i.e $\frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta$; $\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$

i.e $\cot \phi_1 = -\cot 2\theta$; $\cot \phi_2 = \tan 2\theta$

i.e $\cot \phi_1 = \cot(-2\theta)$; $\cot \phi_2 = \cot(\pi/2 - 2\theta)$

$\Rightarrow \phi_1 = -2\theta$; $\phi_2 = \pi/2 - 2\theta$

$\therefore |\phi_1 - \phi_2| = |-2\theta - \pi/2 + 2\theta| = \pi/2$

Thus the curves intersect each other orthogonally.

13) $r = 4\sec^2(\theta/2)$; $r = 9\,\mathrm{cosec}^2(\theta/2)$

$\Rightarrow \log r = \log 4 + 2\log\sec(\theta/2)$; $\log r = \log 9 + 2\log \mathrm{cosec}(\theta/2)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{2}{\sec(\theta/2)} \sec(\theta/2)\tan(\theta/2)\cdot\frac{1}{2}$; $\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\,\mathrm{cosec}(\theta/2)\cot(\theta/2)}{\mathrm{cosec}(\theta/2)}\cdot\frac{1}{2}$

i.e $\frac{1}{r}\frac{dr}{d\theta} = \tan\left(\frac{\theta}{2}\right)$; $\frac{1}{r}\frac{dr}{d\theta} = -\cot\left(\frac{\theta}{2}\right)$

i.e $\cot\phi_1 = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$; $\cot\phi_2 = \cot\left(-\frac{\theta}{2}\right)$

$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$; $\phi_2 = -\frac{\theta}{2}$

$\therefore |\phi_1 - \phi_2| = |\pi/2 - \theta/2 + \theta/2| = \pi/2$

Thus the curves intersect each other orthogonally

14) $r = a \cdot e^{\theta}$; $re^{\theta} = b$

$\Rightarrow \log r = \log a + \theta \log e$; $\log r + \theta \log e = \log b$

But $\log e = 1$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1$; $\frac{1}{r}\frac{dr}{d\theta} + 1 = 0$

i.e $\cot\phi_1 = 1$; $\cot\phi_2 = -1$

$\Rightarrow \phi_1 = \frac{\pi}{4}$; $\phi_2 = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\therefore |\phi_1 - \phi_2| = |\pi/4 + \pi/4| = \pi/2$

Thus the curves intersect each other ortho-gonally.

Find the angle between the following pairs of curves.

15) $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

16) $r = a\log\theta$ and $r = a/\log\theta$

17) $r^2 \sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$

18) $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

19) $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$

20) $r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$

21) $r = a(1+\cos\theta)$ and $r^2 = a^2\cos 2\theta$

22) $r = a\theta/1+\theta$ and $r = a/1+\theta^2$

23) $r = a\theta$ and $r = a/\theta$

Solutions :→

15) $r = \sin\theta + \cos\theta \quad : \quad r = 2\sin\theta$

$\Rightarrow \log r = \log(\sin\theta + \cos\theta) \; : \; \log r = \log 2 + \log(\sin\theta)$

Diff these w.r.to θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} \; ; \; \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

i.e $\cot\phi_1 = \dfrac{\cos\theta(1-\tan\theta)}{\cos\theta(1+\tan\theta)} \; ; \; \cot\phi_2 = \cot\theta$ — $\phi_2 = \theta$

i.e $\cot\phi_1 = \cot\left(\dfrac{\pi}{4} + \theta\right) \Rightarrow \phi_1 = \dfrac{\pi}{4} + \theta$

$\therefore |\phi_1 - \phi_2| = |\pi/4 + \theta - \theta| = \pi/4$

Thus the angle of intersection is $\pi/4$

16) $r = a\log\theta \qquad r = a/\log\theta$

$\Rightarrow \log r = \log a + \log(\log\theta) : \log r = \log a - \log(\log\theta)$

Diff w.r.to θ we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\log\theta \cdot \theta} \; ; \; \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta \cdot \theta}$$

i.e $\cot\phi_1 = \dfrac{1}{\theta\log\theta} \; : \; \cot\phi_2 = \dfrac{-1}{\theta\log\theta}$

Note : we cannot find ϕ_1 and ϕ_2 Explicitly

$\therefore \tan\phi_1 = \theta\log\theta \; : \; \tan\phi_2 = -\theta\log\theta$

Now consider,

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}$$

i.e $\tan(\phi_1 - \phi_2) = \dfrac{2\theta\log\theta}{1-(\theta\log\theta)^2}$ — ①

we have to find θ by solving the given pair of Equations

$r = a\log\theta$ and $r = a/\log\theta$

Equating the RHS we have,

$$a\log\theta = \frac{a}{\log\theta}$$

i.e $(\log\theta)^2 = 1$ or $\log\theta = 1 \Rightarrow \theta = e$

Substituting $\theta = e$ in ① we get

$\tan(\phi_1 - \phi_2) = \dfrac{2e}{1-e^2} \quad (\because \log e = 1)$

Thus the angle of intersection :

$$(\phi_1 - \phi_2) = \tan^{-1}\left(\frac{2e}{1-e^2}\right) = 2\tan^{-1}e$$

17) $r^2\sin 2\theta = 4 \quad : \quad r^2 = 16\sin 2\theta$

$\Rightarrow 2\log r + \log(\sin 2\theta) = \log 4 \; : \; 2\log r = \log 16 + \log\sin 2\theta$

Diff w.r.to θ we get

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0 \; : \; \frac{2}{r}\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

i.e $\dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot 2\theta \; : \; \dfrac{1}{r}\dfrac{dr}{d\theta} = \cot 2\theta$

i.e $\cot\phi_1 = \cot(-2\theta) \; : \; \cot\phi_2 = \cot(2\theta)$

$\Rightarrow \phi_1 = -2\theta \; : \; \phi_2 = 2\theta$

$\therefore |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$ — ①

Now consider,

$r^2 = \dfrac{4}{\sin 2\theta}$ and $r^2 = 16\sin 2\theta$

$\therefore \dfrac{4}{\sin 2\theta} = 16\sin 2\theta$ or $4\sin^2 2\theta = 1$

i.e $\sin^2 2\theta = \dfrac{1}{4}$ or $\sin 2\theta = \dfrac{1}{2} \Rightarrow 2\theta = \dfrac{\pi}{6} \quad \theta = \dfrac{\pi}{12}$

Substituting $\theta = \frac{\pi}{12}$ in (1) we get $|\phi_1 - \phi_2| = \frac{\pi}{3}$

Thus the angle of intersection is $\frac{\pi}{3} = 60^\circ$

18) $r = a(1-\cos\theta)$: $r = 2a\cos\theta$

$\Rightarrow \log r = \log a + \log(1-\cos\theta)$: $\log r = \log 2a + \log(\cos\theta)$

Diff these w.r. to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$: $\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta}$

i.e $\cot\phi_1 = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$: $\cot\phi_2 = -\tan\theta$

i.e $\cot\phi_1 = \cot\left(\frac{\theta}{2}\right)$: $\cot\phi_2 = \cot\left(\frac{\pi}{2}+\theta\right)$

$\Rightarrow \phi_1 = \theta/2$ $\quad \phi_2 = \pi/2 + \theta$

$\therefore |\phi_1 - \phi_2| = |\theta/2 - \pi/2 - \theta| = \pi/2 + \theta/2$ ——— (1)

Now consider, $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$\therefore a(1-\cos\theta) = 2a\cos\theta$

or $3\cos\theta = 1$ or $\theta = \cos^{-1}(1/3)$

or $\theta/2 = 1/2 \cdot \cos^{-1}(1/3)$

we shall substitute this value in (1)

Thus the angle of intersection is

$\frac{\pi}{2} + \frac{1}{2}\cdot\cos^{-1}\left(\frac{1}{3}\right)$

19) $r = 6\cos\theta$: $r = 2(1+\cos\theta)$

$\Rightarrow \log r = \log 6 + \log(\cos\theta)$: $\log r = \log 2 + \log(1+\cos\theta)$

Diff w.r to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta}$; $\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$3\cos\theta = 1 + \cos\theta$

$3\cos\theta - \cos\theta = 1$

$\cos\theta = \frac{1}{2}$

i.e $\cot\phi_1 = -\tan\theta$: $\cot\phi_2 = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$

i.e $\cot\phi_1 = \cot(\pi/2 + \theta)$: $\cot\phi_2 = -\tan(\theta/2)$

$= \cot(\pi/2 + \theta/2)$

$\Rightarrow \phi_1 = \pi/2 + \theta$: $\phi_2 = \pi/2 + \theta/2$

$\therefore |\phi_1 - \phi_2| = \theta/2$ ——— (1)

Equating the RHS of the given Eq^ns we have

$3\cancel{6}\cos\theta = \cancel{2}(1+\cos\theta)$ or $\cos\theta = 1/2 \Rightarrow \theta = \pi/3$

$\therefore$ from (1)

$|\phi_1 - \phi_2| = \pi/6 = 30^\circ$

Thus the angle of intersection is $\pi/6 = 30^\circ$

20) $r^n = a^n \sec(n\theta + \alpha)$ $\quad r^n = b^n \sec(n\theta + \beta)$

$\Rightarrow n\log r = n\log a + \log\sec(n\theta+\alpha)$:

$n\log r = n\log b + \log\sec(n\theta+\beta)$

Diff w.r. to θ we get

$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = \frac{\cancel{n}\cancel{\sec(n\theta}+\alpha)\tan(n\theta+\alpha)}{\cancel{\sec(n\theta}+\alpha)}$:

$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = \frac{\cancel{n}\cancel{\sec(n\theta}+\beta)\tan(n\theta+\beta)}{\cancel{\sec(n\theta}+\beta)}$

i.e $\frac{1}{r}\frac{dr}{d\theta} = \tan(n\theta+\alpha)$: $\frac{1}{r}\frac{dr}{d\theta} = \tan(n\theta+\beta)$

i.e $\cot\phi_1 = \cot[\pi/2 - (n\theta+\alpha)]$: $\cot\phi_2 = \cot[\pi/2 - (n\theta+\beta)]$

$\Rightarrow \phi_1 = \pi/2 - n\theta - \alpha$: $\phi_2 = \pi/2 - n\theta - \beta$

$\therefore |\phi_1 - \phi_2| = |-\alpha + \beta| = \alpha - \beta$

where $\alpha > \beta$

Thus the angle of intersection is

$\alpha - \beta$, where $\alpha > \beta$

21) $r = a(1+\cos\theta)$; $r^2 = a^2\cos 2\theta$

$\Rightarrow \log r = \log a + \log(1+\cos\theta)$; $2\log r = 2\log a + \log(\cos 2\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$; $\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$

i.e $\cot\phi_1 = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$; $\cot\phi_2 = -\tan 2\theta$

i.e $\cot\phi_1 = -\tan(\theta/2)$; $\cot\phi_2 = \cot(\pi/2 + 2\theta)$

i.e $\cot\phi_1 = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$; $\phi_2 = \frac{\pi}{2} + 2\theta$

$\Rightarrow \phi_1 = \pi/2 + \theta/2$

$\therefore |\phi_1 - \phi_2| = |\pi/2 + \theta/2 - \pi/2 - 2\theta| = 3\theta/2$ —— ①

Now squaring the first of the given Equations and then Equating with the RHS of the second Equation, we have

$a^2(1+\cos\theta)^2 = a^2\cos 2\theta$

i.e $1 + 2\cos\theta + \cos^2\theta = 2\cos^2\theta - 1$

or $\cos^2\theta - 2\cos\theta - 2 = 0$

$\therefore \cos\theta = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

Since $\cos\theta$ cannot exceed 1 numerically we have to take

$\cos\theta = 1 - \sqrt{3}$

i.e $1 - 2\sin^2(\theta/2) = 1 - \sqrt{3}$ or $\sin^2(\theta/2) = \sqrt{3}/2 = \sqrt{3/4}$

i.e $\sin(\theta/2) = (\sqrt{3/4})^{1/2} = (3/4)^{1/4}$

$\therefore \theta/2 = \sin^{-1}(3/4)^{1/4}$ and we shall substitute this value in ①

Thus the angle of intersection is $3\sin^{-1}(3/4)^{1/4}$

22) $r = a\theta/1+\theta$; $r = a/1+\theta^2$

$\Rightarrow \log r = \log a + \log\theta - \log(1+\theta)$; $\log r = \log a - \log(1+\theta^2)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} - \frac{1}{1+\theta}$; $\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\theta}{1+\theta^2}$

i.e $\cot\phi_1 = \frac{1}{\theta(1+\theta)}$; $\cot\phi_2 = \frac{-2\theta}{1+\theta^2}$

$\Rightarrow \tan\phi_1 = \theta + \theta^2$; $\tan\phi_2 = \frac{1+\theta^2}{-2\theta}$

Also by equating the RHS of the given Equations we have

$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$

or $\theta + \theta^3 = 1 + \theta$ or $\theta^3 = 1 \Rightarrow \theta = 1$

$\therefore \tan\phi_1 = 2$ and $\tan\phi_2 = -1$ at $\theta = 1$

Consider,

$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\tan\phi_2}$

$\therefore \tan(\phi_1 - \phi_2) = \frac{2-(-1)}{1+(-2)} = -3$

Thus by taking the absolute value, the angle of intersection is $\tan^{-1}(3)$

23) $r = a\theta$; $r = a/\theta$

$\Rightarrow \log r = \log a + \log\theta$; $\log r = \log a - \log\theta$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta}$; $\frac{1}{r}\frac{dr}{d\theta} = \frac{-1}{\theta}$

i.e $\cot\phi_1 = \frac{1}{\theta}$; $\cot\phi_2 = -\frac{1}{\theta}$

or $\tan\phi_1 = \theta$; $\tan\phi_2 = -\theta$

Also by Equating the RHS of the given Equations we have

$a\theta = a/\theta$ or $\theta^2 = 1 \Rightarrow \theta = \pm 1$

when $\theta = 1$, $\tan\phi_1 = 1$, $\tan\phi_2 = -1$ and

when $\theta = -1$, $\tan\phi_1 = -1$, $\tan\phi_2 = 1$

$\therefore \tan\phi_1 \cdot \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$

Thus the angle of intersection is $\pi/2$

⇒ Find the angle between the curves $r = a(1+\sin\theta)$ and $r = a(1-\cos\theta)$.

sol^n: By problem 8 and 9

we have $\phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$

and $\phi_2 = \frac{\theta}{2}$ respectively for the given two curves

Thus the required angle between the curves is $|\phi_1 - \phi_2| = \pi/4$

Pedal Equation of a polar curve :⇒

In the context of deriving an Expression for the length of the perpendicular (p) from the pole to the tangent we obtained the Expression in the form $p = r\sin\phi$

The Equation of the given curve $r = f(\theta)$ expressed interms of p and r is called as the pedal Equation or p-r Equation of a curve $r = f(\theta)$

Many Equations of the Standard Cartesian curves $y = f(x)$ are expressible in the parametric form $x = x(t)$, $y = y(t)$, Eliminating t we get $y = f(x)$. We have a similar concept in the case of polar curve $r = f(\theta)$

Note :⇒ If we are unable to obtain ϕ Explicitly interms of θ, we have to square and take the reciprocal of $p = r\sin\phi$

This will give us

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$$

We substitute for $\cot\phi$ itself interms of θ. Elimination of θ by using the given Equation will give us the pedal Equation

Problems :→

24) Find the pedal Equation of the following curves.

24) $2a/r = (1+\cos\theta)$

25) $r(1-\cos\theta) = 2a$

26) $r^2 = a^2 \sec 2\theta$

27) $r^n = a^n \cos n\theta$

28) $r^m = a^m(\cos m\theta + \sin m\theta)$

29) $r = a(1+\cos\theta)$

30) $1/r = 1 + e\cos\theta$

31) $r^n = a^n \cdot \operatorname{sech} n\theta$

Solutions :→

24) $\frac{2a}{r} = (1+\cos\theta)$

$\Rightarrow \log 2a - \log r = \log(1+\cos\theta)$

Diff w.r.to θ. we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

$$= -\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$= -\tan(\theta/2)$$

i.e $\cot\phi = \cot(\pi/2 + \theta/2)$

$$\phi = \pi/2 + \theta/2$$

Consider $p = r\sin\phi$ and substituting the value of ϕ we have,

At $p = r\sin(\pi/2 + \theta/2) = r\cos(\theta/2)$

Now we have

$$\frac{2a}{r} = 1+\cos\theta \quad \text{——} \ ①$$

$$p = r\cos(\theta/2) \quad \text{——} \ ②$$

we have to eliminate θ from ① and ②

(1) can be put in the form

$$\frac{\cancel{2}a}{r} = \cancel{2}\cos^2(\theta/2) \text{ or } \frac{a}{r} = \cos^2\left(\frac{\theta}{2}\right)$$

Also from ②, $\frac{p}{r} = \cos\left(\frac{\theta}{2}\right)$

Hence we get, $\frac{a}{r} = \left(\frac{p}{r}\right)^2$ or $\frac{a}{r} = \frac{p^2}{r^2}$ or $p^2 = ar$

Thus $p^2 = ar$ is the required pedal Equation

25) $r(1-\cos\theta) = 2a$

$\Rightarrow \log r + \log(1-\cos\theta) = \log 2a$

Diff w.r.to θ. we get

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \text{ or } \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

i.e $\cot\phi = -\frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)}$

$$= -\cot(\theta/2)$$

i.e $\cot\phi = \cot(-\theta/2) \Rightarrow \phi = -(\theta/2)$

Consider, $p = r\sin\phi$

$\therefore p = r\sin(-\theta/2)$ or $p = -r\sin(\theta/2)$

Now we have,

$$r(1-\cos\theta) = 2a \quad \text{——} \ ①$$

$$p = -r\sin(\theta/2) \quad \text{——} \ ②$$

we have to eliminate θ from ① & ②

① Can be put in the form

$$r\cdot 2\sin^2(\theta/2) = 2a \text{ or } r\sin^2\left(\frac{\theta}{2}\right) = a$$

$\therefore r\left(\frac{p^2}{r^2}\right) = a$ or $p^2 = ar$

Since $\frac{p}{r} = \sin\left(\frac{\theta}{2}\right)$ from (2)

Thus $p^2 = ar$ is the required pedal Equation

26) $r^2 = a^2 \sec 2\theta$.

$\Rightarrow 2\log r = 2\log a + \log(\sec 2\theta)$

Diff w.r.to θ we get

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sec 2\theta \cdot \tan 2\theta}{\sec 2\theta}$$

or $\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$.

i.e $\cot\phi = \cot(\pi/2 - 2\theta)$

$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$

consider, $p = r\sin\phi$

$\therefore p = r\sin\left(\frac{\pi}{2} - 2\theta\right)$ or $p = r\cos 2\theta$

Now we have,

$r^2 = a^2\sec 2\theta$ —— (1)

$p = r\cos 2\theta$ —— (2)

From (2), $\frac{p}{r} = \cos 2\theta$ or $\frac{r}{p} = \sec 2\theta$

Substituting in (1) we get

$r^2 = a^2\left(\frac{r}{p}\right)$ or $pr = a^2$

Thus $pr = a^2$ is the required pedal Equation

27) $r^n = a^n\cos n\theta$

$\Rightarrow n\log r = n\log a + \log(\cos n\theta)$

Diff w.r.to θ we get

$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta}$ or $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$.

i.e $\cot\phi = \cot\left(\frac{\pi}{2} + n\theta\right) \rightarrow \phi = \frac{\pi}{2} + n\theta$

Consider, $p = r\sin\phi$

$\therefore p = r\sin(\pi/2 + n\theta)$ or $p = r\cos n\theta$

Now we have,

$r^n = a^n\cos n\theta$ —— (1)

$p = r\cos n\theta$ —— (2)

Hence, (1) as a consequence of (2) is

$r^n = a^n(p/r)$

Thus $r^{n+1} = pa^n$ is the required pedal Eq^n

28) $r^m = a^m(\cos m\theta + \sin m\theta)$

$\Rightarrow m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Diff w.r.to θ we get

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m\sin m\theta + m\cos m\theta}{\cos m\theta + \sin m\theta}$$

i.e $\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)}$

i.e $\cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \phi = \frac{\pi}{4} + m\theta$.

consider, $p = r\sin\phi$

$\therefore p = r\sin(\pi/4 + m\theta)$

i.e $p = r[\sin(\pi/4)\cos m\theta + \cos(\pi/4)\sin m\theta]$

sin(A+B) formula

i.e $p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$

$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Now we have $r^m = a^m(\cos m\theta + \sin m\theta)$ —— ①

$p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$ —— ②

using ② in ① we get

$r^m = a^m \cdot \frac{p\sqrt{2}}{r}$ or $r^{m+1} = \sqrt{2}\, a^m . p$

Thus $r^{m+1} = \sqrt{2} \cdot a^m . p$ is the required pedal eqⁿ

29) $r = a(1+\cos\theta)$

$\Rightarrow \log r = \log a + \log(1+\cos\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$

$= -\tan(\theta/2)$

i.e $\cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$

Consider, $p = r\sin\phi$

$\therefore p = r\sin(\pi/2 + \theta/2)$ or $p = r\cos(\theta/2)$

Now we have,

$r = a(1+\cos\theta)$ —— ①

$p = r\cos(\theta/2)$ —— ②

① can be put in the form, $r = a\, 2\cos^2(\theta/2)$

i.e, $r = 2a\cos^2(\theta/2)$

From (2), $p/r = \cos(\theta/2)$ and hence ① becomes

$r = 2a \cdot (p^2/r^2)$ or $r^3 = 2a \cdot p^2$

Thus $r^3 = 2ap^2$ is the required pedal eqⁿ

30) $l/r = 1 + e\cos\theta$

$\Rightarrow \log l - \log r = \log(1+e\cos\theta)$

Diff w.r.to θ we get

$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$

i.e $\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$

we cannot find ϕ explicitly

Consider, $p = r\sin\phi$

By squaring and taking the reciprocal we have,

$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$ or $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

Substituting for $\cot\phi$ itself we have

$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right\}$ —— ①

Also we have, $\frac{l}{r} = 1 + e\cos\theta$ —— ②

we need to eliminate θ from ① and ②

from ② $\frac{l}{r} - 1 = e\cos\theta$ —— ③

Also, $e^2\sin^2\theta = e^2(1-\cos^2\theta) = e^2 - e^2\cos^2\theta$

By using ③ we have,

$e^2\sin^2\theta = e^2 - \left(\frac{l}{r} - 1\right)^2$ —— ④

Now substituting ③ and ④ in ① we have

$\frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{e^2 - \left(\frac{l}{r} - 1\right)^2}{(l^2/r^2)}\right\}$

$\frac{d}{dr}$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left\{e^2 - \left(\frac{l}{r} - 1\right)^2\right\}$$

i.e $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left\{e^2 - \frac{l^2}{r^2} + \frac{2l}{r} - 1\right\}$$

i.e $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{r^2} + \frac{2}{lr} - \frac{1}{l^2}$$

Thus $\frac{1}{p^2} = \frac{e^2-1}{l^2} + \frac{2}{lr}$ is the required pedal Equation.

31) $r^n = a^n \operatorname{sech} n\theta$

$\Rightarrow n \log r = n \log a + \log(\operatorname{sech} n\theta)$

Diff w.r.to θ we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \operatorname{sech} n\theta \cdot \tanh n\theta}{\operatorname{sech} n\theta}$$

i.e $\frac{1}{r}\frac{dr}{d\theta} = -\tanh n\theta$

i.e $\cot\phi = -\tanh n\theta$ and ϕ cannot be found explicitly

Consider, $p = r\sin\phi$

Squaring and taking the reciprocal we get

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

i.e $\frac{1}{p^2} = \frac{1}{r^2}(1+\tanh^2 n\theta)$ —— (1)

Also we have, $r^n = a^n \operatorname{sech} n\theta$

$\therefore \frac{r^n}{a^n} = \operatorname{sech} n\theta$ and we have

$1 - \tanh^2 n\theta = \operatorname{sech}^2 n\theta$

$\therefore \tanh^2 n\theta = 1 - \operatorname{sech}^2 n\theta = 1 - \left(\frac{r^n}{a^n}\right)^2$

we substitute this Expression in the RHS of (1)

Thus $\frac{1}{p^2} = \frac{1}{r^2}\left\{2 - \frac{r^{2n}}{a^{2n}}\right\}$ is the required pedal Equation.

32) For the Equiangular Spiral $r = ae^{\theta\cot\alpha}$ a and α are constants show that the tangent is inclined at a constant angle with the radius vector and hence find the pedal Equation of the curve

Sol^n:$\Rightarrow$ We have $r = ae^{\theta\cot\alpha}$

$\Rightarrow \log r = \log a + \theta\cot\alpha \log e$. But $\log e = 1$

$\log r = \log a + \cot\alpha \cdot \theta$

Diff w.r.to θ we get

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \cot\alpha \cdot 1$$

i.e $\cot\phi = \cot\alpha \Rightarrow \phi = \alpha = $ constant

Thus the tangent is inclined at a constant angle with the radius vector

Consider, $p = r\sin\phi$. But $\phi = \alpha$.

$\therefore p = r\sin\alpha$ this being independent of θ is the pedal Equation

Thus, $p = r\sin\alpha$ is a the required pedal Equation

33) Show that for the curve $r\cos(\sqrt{a^2-b^2}/a)\theta = \sqrt{a^2-b^2}$, $p^2(r^2+b^2) = a^2r^2$

Sol^n :→ we have $r\cos(\sqrt{a^2-b^2}/a)\theta = \sqrt{a^2-b^2}$

For convenience, let $\sqrt{a^2-b^2}/a = k$, a constant

we now have, $r\cos k\theta = k.a$

$\Rightarrow \log r + \log(\cos k\theta) = \log(ka)$

Diff w.r.to θ we get

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{-k\sin k\theta}{\cos k\theta} = 0$$

i.e $\cot\phi = k\tan k\theta$

we cannot find ϕ Explicitly

Consider $p = r\sin\phi$

squaring and taking the reciprocal we have

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

i.e $\frac{1}{p^2} = \frac{1}{r^2}(1+k^2\tan^2 k\theta)$ —— ①

Also $r\cos k\theta = ka$ —— ②

We need to eliminate θ from ① and ②

From ② $\cos k\theta = \frac{ka}{r} \Rightarrow \sec k\theta = \frac{r}{ka}$

Now, $\tan^2 k\theta = \sec^2 k\theta - 1 = \frac{r^2}{k^2a^2} - 1$

Substituting this expression in ① we get

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+k^2\left(\frac{r^2}{k^2a^2}-1\right)\right\}$$

i.e $\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{r^2}{a^2}-k^2\right\}$ But $k^2 = \frac{a^2-b^2}{a^2}$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{r^2}{a^2}-\frac{a^2-b^2}{a^2}\right\}$$

i.e, $\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^2+r^2-a^2+b^2}{a}\right\}$

or $\frac{1}{p^2} = \frac{r^2+b^2}{r^2a^2}$

Thus $p^2(r^2+b^2) = a^2r^2$ as required

34) Establish the pedal Equation of the curve $r^n = a^n\sin n\theta + b^n\cos n\theta$ in the form $p^2(a^{2n}+b^{2n}) = r^{2n+2}$

Sol^n :→ we have, $r^n = a^n\sin n\theta + b^n\cos n\theta$

$\Rightarrow n\log r = \log(a^n\sin n\theta + b^n\cos n\theta)$

Diff w.r.to θ we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{na^n\cos n\theta - nb^n\sin n\theta}{a^n\sin n\theta + b^n\cos n\theta}$$

Dividing by n,

$$\cot\phi = \frac{a^n\cos n\theta - b^n\sin n\theta}{a^n\sin n\theta + b^n\cos n\theta}$$

Consider, $p = r\sin\phi$

Since ϕ cannot be found,

Squaring and taking the reciprocal we get

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{(a^n\cos n\theta - b^n\sin n\theta)^2}{(a^n\sin n\theta + b^n\cos n\theta)^2}\right\}$$

i.e, $\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$

i.e $\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^{2n}(\sin^2 n\theta + \cos^2 n\theta) + b^{2n}(\cos^2 n\theta + \sin^2 n\theta)}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$

product terms cancels out in the Nr

i.e $\frac{1}{p^2} = \frac{1}{r^2} \frac{a^{2n}+b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}$

or $\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{a^{2n}+b^{2n}}{(r^n)^2}$

(by using the given Equation)

Thus $p^2(a^{2n}+b^{2n}) = r^{2n+2}$ is the required pedal equation

35) Find the length of the perpendicular from the pole to the tangent at the point $(a, \pi/2)$ on the curve $r = a(1-\cos\theta)$

Sol^n :→ we have $r = a(1-\cos\theta)$

$\Rightarrow \log r = \log a + \log(1-\cos\theta)$

Diff w.r.to θ we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$= \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2(\theta/2)}$

$= \cot(\theta/2)$

i.e $\cot\phi = \cot(\theta/2)$

$\Rightarrow \phi = \theta/2$

Length of the perpendicular, $p = r\sin\phi$

i.e $p = r\sin(\theta/2)$

Substituting $(r, \theta) = (a, \pi/2)$

we get $p = a\sin(\pi/4) = a/\sqrt{2}$

Thus $p = \frac{a}{\sqrt{2}}$

Assignment

Find the angle between the radius vector and the tangent for the following curves

1) $r\sec^2(\theta/2) = 2a$ 2) $r = a\,\mathrm{cosec}^2(\theta/2)$

3) $r^2 = a^2(\cos 2\theta + \sin 2\theta)$ 4) $r^n \mathrm{cosec}\, n\theta = a^n$

S.T the following pairs of curves intersect each other orthogonally

5) $r\sec^2(\theta/2) = a$ and $r\,\mathrm{cosec}^2(\theta/2) = b$

6) $r^n \cos n\theta = a^n$ and $r^n \sin n\theta = b^n$

7) $2a/r = 1+\cos\theta$ and $2a/r = 1-\cos\theta$

8) $r^2 = a^2\cos 2\theta$ and $r^2 = a^n \sin 2\theta$

Find the angle of intersection for the following pairs of curves.

9) $r^n = a^n(\sin n\theta + \cos n\theta)$ and $r^n = a^n \sin n\theta$

10) $r^2\cos(2\theta+\alpha) = a^2$ and $r^2\cos(2\theta+\beta) = b^2$

Obtain the pedal Equation of the following curves

11) $r^2 \cos 2\theta = a^2$ 12) $r = 2a/1 + \cos\theta$

13) $r^2 = a^2 \sin 2\theta + b^2 \cos 2\theta$ 14) $r^n \sec n\theta = a^n$

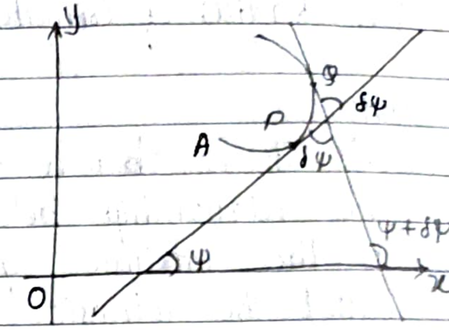
15) Show that for the curve $r \sin^2(\theta/2) = a$ the length of the perpendicular from the pole to the tangent at the point $(2a, \pi/2)$ on the curve is equal to $a\sqrt{2}$

16) Show that the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/6$ on the curve $r^2 \cos 2\theta = a^2$ is equal to $a/\sqrt{2}$.

$\frac{a}{\sqrt{2}}$ $\left(a, \frac{\pi}{6}\right)$

Curvature and Radius of curvature

Defn:→



consider a curve in the xoy plane and let A be a fixed point on it. Let P and Q be two neighbouring points on the curve such that

$\widehat{AP} = s$ and $\widehat{AQ} = s + \delta s$ so that $\widehat{PQ} = \delta s$

Let ψ and $\psi + \delta\psi$ respectively be the angles made by the tangent at P and Q with the x-axis. The angle $\delta\psi$ between the tangents is called the bending of the curve which depends on δs. $\frac{\delta\psi}{\delta s}$ is called as the mean curvature of the arc PQ. Also the amount of bending of the curve at P is called as the curvature of the curve at P and is defined mathematically as

$\lim_{\substack{\delta s \to 0 \\ (Q \to P}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$ be denoted by K.

That is, curvature $= K = \frac{d\psi}{ds}$ Further if $K \neq 0$

the reciprocal of the curvature is called as the radius of curvature ~~is called~~ & is denoted by ρ

i.e. Radius of Curvature $= \rho = \frac{1}{K} = \frac{ds}{d\psi}$

Note :-

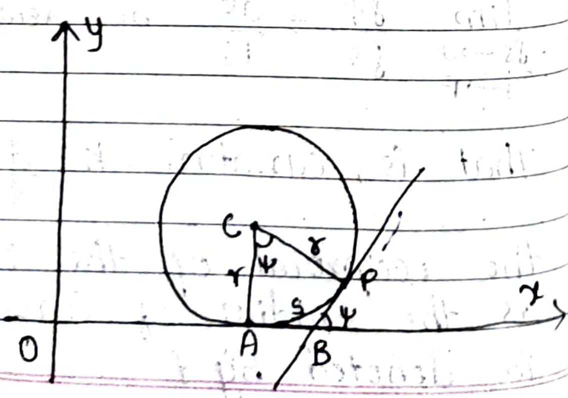
1) As it is obvious that ψ depends on s, the relationship between these is called as the intrinsic equation and (s, ψ) are called the intrinsic coordinates of the point P.

2) We always take the sign of K and ρ to be positive.

Remark :-

Curvature being the amount of bending is obviously zero for a straight line at all the points on it. It is easy to visualize that the circle has an uniform bending and hence the curvature of a circle is a constant which will be established mathematically.

prove that the curvature of a circle is a constant.



Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and $P(x,y)$ be any point on the circle such that $\widehat{AP} = s$. Let ψ be the angle made by the tangent at P with the x-axis at the point B. (interior angle being $\pi - \psi$)

Clearly $CA = CP = r$ = radius.

We have from the quadrilateral $CABP$.

$\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$

i.e. $\hat{C} + \frac{\pi}{2} + (\pi - \psi) + \frac{\pi}{2} = 2\pi$

Hence, $A\hat{C}P = \psi$.

We have a known result.

$s = r\psi$ or $\psi = \frac{s}{r}$

$\therefore \frac{d\psi}{ds} = \frac{1}{r}$ = constant

Thus the curvature, $K = 1/r$ = constant.

Expression for the radius of curvature for a Cartesian curve $y = f(x)$.

Denoting, $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

the Expression for ρ can be proved in the following form

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Note :⇒

Sometimes y_1 at some point on the curve becomes infinity (when the tangent is perpendicular to the x-axis, $\tan\psi = \tan 90° = \infty$) in which case we cannot apply the formula for ρ in this form.

In such a case we have to use the formula in the following alternative form

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy} \ \& \ x_2 = \frac{d^2x}{dy^2}$$

Problems :⇒

1) Find the radius of curvature for the curve whose intrinsic equation is

$s = a \log \tan(\pi/4 + \psi/2)$

Sol^n :⇒ $s = a \log \tan(\pi/4 + \psi/2)$ and we have $\rho = \frac{ds}{d\psi}$

Diff w.r.t ψ we have

$$\frac{ds}{d\psi} = a \cdot \frac{1}{\tan(\pi/4+\psi/2)} \cdot \sec^2(\pi/4+\psi/2) \cdot \frac{1}{2}$$

$$= \frac{a}{2} \frac{\cos(\pi/4+\psi/2)}{\sin(\pi/4+\psi/2)} \cdot \frac{1}{\cos^2(\pi/4+\psi/2)}$$

$$= \frac{a}{2\sin(\pi/4+\psi/2)\cdot\cos(\pi/4+\psi/2)}$$

But, $2\sin\theta\cdot\cos\theta = \sin 2\theta$

$$\therefore \frac{ds}{d\psi} = \frac{a}{\sin[2(\pi/4+\psi/2)]}$$

$$= \frac{a}{\sin(\pi/2+\psi)} = \frac{a}{\cos\psi} = a\sec\psi$$

Thus $\rho = a\sec\psi$

2) Show that the radius of curvature for the catenary of uniform strength $y = a \log \sec(x/a)$ is $a\sec(x/a)$

Sol^n :⇒ We have, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Consider, $y = a\log\sec(x/a)$

$$\therefore \frac{dy}{dx} = y_1 = \frac{a}{\sec(x/a)} \cdot \sec(x/a)\cdot\tan(x/a)\cdot\frac{1}{a}$$

i.e $y_1 = \tan(x/a)$. Also $y_2 = \frac{1}{a}\sec^2(x/a)$

$$\text{Hence, } \rho = \frac{[1+\tan^2(x/a)]^{3/2} \cdot a}{\sec^2(x/a)}$$

$$= \frac{a[\sec^2(x/a)]^{3/2}}{\sec^2(x/a)}$$

i.e $\rho = \frac{a\sec^3(x/a)}{\sec^2(x/a)} = a\sec(x/a)$

Thus. $\rho = a\sec(x/a)$

3) Show that for the catenary $y = c\cosh(x/c)$ the radius of curvature is equal to y^2/c which is also equal to the length of the normal intercepted between the curve and the x axis

Sol^n:⇒ we have, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$y = c\cosh(x/c)$ by data

$\therefore y_1 = c\cdot\sinh(x/c)\cdot\frac{1}{c} = \sinh(x/c)$

$y_2 = \frac{1}{c}\cosh(x/c)$

Hence, $\rho = \frac{[1+\sinh^2(x/c)]^{3/2}\cdot c}{\cosh(x/c)}$

$= \frac{c[\cosh^2(x/c)]^{3/2}}{\cosh(x/c)}$

i.e $\rho = \frac{c\cdot\cosh^3(x/c)}{\cosh(x/c)} = c\cdot\cosh^2(x/c)$

But $y/c = \cosh(x/c)$ and hence $\rho = c\left(\frac{y^2}{c^2}\right) = \frac{y^2}{c}$

Also w.k.t the length of the normal l is $y\sqrt{1+y_1^2}$

$\therefore l = c\cosh(x/c)\sqrt{1+\sinh^2(x/c)}$

$= c\cdot\cosh^2(x/c) = y^2/c$

Thus the required results are proved

$\rho = y^2/c = l$.

4) Find the radius of curvature for the curve $y = ax^2+bx+c$ at $x = \frac{1}{2a}[\sqrt{a^2-1}-b]$

Sol^n:⇒ $y = ax^2+bx+c$ by data

$\therefore y_1 = 2ax+b,\ y_2 = 2a$

At the given point,

$y_1 = 2a\cdot\frac{1}{2a}[\sqrt{a^2-1}-b]+b = \sqrt{a^2-1}$ and $y_2 = 2a$

we have, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{[1+(a^2-1)]^{3/2}}{2a} = \frac{(a^2)^{3/2}}{2a} = \frac{a^2}{2}$

Thus, $\rho = a^2/2$

5) Find the radius of curvature for the Folium of DesCartes $x^3+y^3 = 3a\cdot xy$ at the point $(3a/2, 3a/2)$ on it

Sol^n:⇒ $x^3+y^3 = 3a\cdot xy$, by data

Diff w.r.t x we have

$3x^2+3y^2\frac{dy}{dx} = 3a\left(x\cdot\frac{dy}{dx}+y\right)$

i.e $3(y^2-ax)\frac{dy}{dx} = 3(ay-x^2)$ or

$\frac{dy}{dx} = y_1 = \frac{ay-x^2}{y^2-ax}$

At $(3a/2, 3a/2)$, $y_1 = \frac{3a^2/2-9a^2/4}{9a^2/4-3a^2/2} = -1$

Next, $\frac{d^2y}{dx^2} = y_2 = \frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$

At $(3a/2, 3a/2)$ we note that

$y^2-ax = 9a^2/4-3a^2/2 = 3a^2/4$ and

$ay-x^2 = 3a^2/2-9a^2/4 = -3a^2/4$

Hence at $(3a/2, 3a/2)$

$y_2 = \frac{(3a^2/4)(-a-3a)-(-3a^2/4)(-3a-a)}{(3a^2/4)^2}$

i.e $y_2 = \frac{-3a^3-3a^3}{9a^4/16} = \frac{16(-6a^3)}{9a^4} = \frac{-32}{3a}$

we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Hence, $\rho = \dfrac{(1+1)^{3/2}}{-32/3a} = \dfrac{2\sqrt{2}\cdot 3a}{-32}$

$= \dfrac{-3\sqrt{2}a}{16} = \dfrac{-3a}{8\sqrt{2}}$

Thus. $|\rho| = \dfrac{3a}{8\sqrt{2}}$

6) Find the radius of curvature for the curve $y^2 = \dfrac{4a^2(2a-x)}{x}$ where the curve meets the x-axis.

Sol^n: If the curve meets the x axis then $y=0$

$\therefore \dfrac{4a^2(2a-x)}{x} = 0 \Rightarrow 4a^2(2a-x) = 0$ or $x = 2a$

Hence $(2a, 0)$ is the point on the curve at which we have to find ρ.

The given Equation can be put in the form.

$y^2 = \dfrac{8a^3}{x} - 4a^2$

Diff w.r.to x we have

$2yy_1 = \dfrac{-8a^3}{x^2}$ or $y_1 = \dfrac{-4a^3}{x^2 y}$

At $(2a, 0)$, y_1 becomes infinity and hence we have to consider dx/dy

Let, $x_1 = \dfrac{dx}{dy} = \dfrac{-x^2 y}{4a^3}$ & $x_1 = 0$ at $(2a, 0)$

Now, $x_2 = \dfrac{d^2x}{dy^2} = \dfrac{-1}{4a^3}\left[x^2\cdot 1 + y\cdot 2xx_1\right]$

$\therefore$ At $(2a, 0)$: $x_2 = -4a^2/4a^3 = -1/a$

we have, $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$\rho = \dfrac{(1+0)^{3/2}}{-1/a} = -a$

Thus, $|\rho| = a$

Find ρ of the curve $y^2 = \dfrac{a^2(a-x)}{x}$ at the point $(a, 0)$.

$\Rightarrow$ Here we have 'a' instead of $2a$

Evidently $\rho = a/2$.

7) Find the radius of curvature for the curve $x^2y = a(x^2+y^2)$ at $(-2a, 2a)$

Sol^n :$\Rightarrow$ consider $x^2y = a(x^2+y^2)$

Diff w.r.to x

$\therefore x^2y_1 + 2xy = 2ax + 2ayy_1$

i.e $y_1(x^2 - 2ay) = 2ax - 2xy$

or $y_1 = \dfrac{2ax-2xy}{x^2-2ay}$; At $(-2a, 2a)$, y_1 is infinity

Hence, $x_1 = \dfrac{dx}{dy} = \dfrac{1}{y_1} = \dfrac{x^2-2ay}{2ax-2xy}$ and at $(-2a, 2a)$

we have $x_1 = 0$

Also, $x_2 = \dfrac{d^2x}{dy^2} = \dfrac{(2ax-2xy)(2xx_1-2a)-(x^2-2ay)(2ax_1-2x-2x_1 \ldots)}{(2ax-2xy)^2}$

we know that at the point $(-2a, 2a)$

$(2ax - 2xy) = 4a^2$ and $(x^2 - 2ay) = 0$

$\therefore (x_2)_{(-2a,2a)} = \dfrac{(4a^2)(-2a)}{16a^4} = \dfrac{-1}{2a}$

we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1)^{3/2}}{-1/2a} = -2a$

Thus, $|\rho| = 2a$

8) Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=4$ at the point where it cuts the line passing through the origin making an angle $45°$ with the x-axis.

Sol^n:- The equation of the line is $y=x$ and we shall find the point of intersection of this line with the curve $\sqrt{x}+\sqrt{y}=4$

This Eq^n when $y=x$ becomes

$\sqrt{x}+\sqrt{x}=4$ or $2\sqrt{x}=4$ or $\sqrt{x}=2$ or $x=4$

$\therefore$ The point of intersection is $(4,4)$

Consider, $\sqrt{x}+\sqrt{y}=4$

Diff w.r.to x

$\therefore \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}} y_1 = 0$ or $\dfrac{y_1}{\sqrt{y}} = \dfrac{-1}{\sqrt{x}}$

i.e $y_1 = -\sqrt{y}/\sqrt{x}$. At $(4,4)$ we get $y_1 = -1$

Now,

$y_2 = \dfrac{d^2y}{dx^2} = \dfrac{\sqrt{x}\cdot -1/2\sqrt{y}\, y_1 - (-\sqrt{y})\cdot 1/2\sqrt{x}}{x}$

$\therefore$ At $(4,4)$, $y_2 = \dfrac{1/2 + 1/2}{4} = \dfrac{1}{4}$

we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(2)^{3/2}}{1/4} = 4\cdot 2\sqrt{2} = 8\sqrt{2}$

Thus, $\rho = 8\sqrt{2}$

9) For the curve $y = ax/a+x$. show that $(2\rho/a)^{2/3} = (x/y)^2 + (y/x)^2$

Sol^n:$\Rightarrow$ $y = \dfrac{ax}{a+x}$, by data

$\therefore y_1 = \dfrac{(a+x)\cdot a - ax\cdot 1}{(a+x)^2} = \dfrac{a^2}{(a+x)^2}$

Also, $y_2 = \dfrac{-2a^2}{(a+x)^3}$

we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Hence, $\rho = \dfrac{\left[1 + \dfrac{a^4}{(a+x)^4}\right]^{3/2} \cdot (a+x)^3}{-2a^2}$

$\rho = \dfrac{[(a+x)^4 + a^4]^{3/2}\cdot (a+x)^3}{-2a^2\{(a+x)^4\}^{3/2}}$

$= \dfrac{[(a+x)^4 + a^4]^{3/2}\cdot (a+x)^3}{-2a^2(a+x)^6}$

or $-2\rho = \dfrac{[(a+x)^4 + a^4]^{3/2}}{a^2(a+x)^3}$

$\Rightarrow (-2\rho)^{2/3} = \dfrac{(a+x)^4 + a^4}{a^{4/3}(a+x)^2}$

we note that $(-2)^{2/3} = 2^{2/3}$

$\therefore (2\rho)^{2/3} = \dfrac{1}{a^{4/3}}\left\{(a+x)^2 + \left(\dfrac{a^2}{a+x}\right)^2\right\}$

But $y = \dfrac{ax}{a+x}$ by data

$\therefore (2\rho)^{2/3} = \dfrac{1}{a^{4/3}}\left\{\dfrac{a^2x^2}{y^2} + \dfrac{a^2y^2}{x^2}\right\}$

$= \dfrac{a^2}{a^{4/3}}\left\{\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2}\right\}$

i.e $(2\rho)^{2/3} = a^{2/3}\left\{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2\right\}$

Thus, $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$

10) Find the radius of curvature of the curve $x = a\log(\sec t + \tan t)$, $y = a\sec t$

Sol^n:→ $x = a\log(\sec t + \tan t)$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t}(\sec t\tan t + \sec^2 t)$$

$$= \frac{a\sec t(\sec t + \tan t)}{\sec t + \tan t}$$

$$\therefore \frac{dx}{dt} = a\sec t$$

Also, $y = a\sec t$ gives $\frac{dy}{dt} = a\sec t\cdot\tan t$

Now, $y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\sec t\cdot\tan t}{a\sec t} = \tan t$

Diff w.r.to x we get

$$y_2 = \sec^2 t\cdot\frac{dt}{dx}$$

$$\therefore y_2 = \sec^2 t\cdot\frac{1}{a\sec t} = \frac{\sec t}{a}$$

we have, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}\cdot a}{\sec t} = \frac{a\sec^3 t}{\sec t} = a\sec^2 t$$

Thus, $\rho = a\sec^2 t$

11) Show that the radius of curvature at any point θ on the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos(\theta/2)$

Sol^n:⇒

$x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1+\cos\theta)$; $\frac{dy}{d\theta} = a\sin\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2\sin\theta/2\cdot\cos\theta/2}{2\cos^2\theta/2}$$

$$y_1 = \tan(\theta/2)$$

Diff w.r.to x we get

$$y_2 = \sec^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\frac{d\theta}{dx}$$

$$= \sec^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$$y_2 = \frac{1}{4a}\sec^4(\theta/2)$$

we have. $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)}$$

$$\rho = \frac{[\sec^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

$$= \frac{4a}{\sec(\theta/2)}$$

Thus, $\rho = 4a\cos(\theta/2)$

12) Find the radius of curvature of the tractrix $x = a[\cos t + \log \tan(t/2)]$, $y = a\sin t$

Sol^n :⇒

For the curve we have

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan(t/2)} \cdot \sec^2(t/2) \cdot \frac{1}{2}\right\}$$

$$= a\left[-\sin t + \frac{1}{2\cos(t/2)\cdot\sin(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$= a \cdot \frac{\cos^2 t}{\sin t}$$

i.e $\frac{dx}{dt} = a\cos^2 t \cdot \operatorname{cosec} t$; Also $\frac{dy}{dt} = a\cos t$

Now, $y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{a\cos t}{a\cos^2 t \cdot \operatorname{cosec} t} = \tan t$$

Further, $y_2 = \sec^2 t \cdot \frac{dt}{dx}$

$$= \frac{\sec^2 t}{a\cos^2 t \cdot \operatorname{cosec} t} = \frac{\sec^4 t \cdot \sin t}{a}$$

we have, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2} \cdot a}{\sec^4 t \cdot \sin t}$$

$$= \frac{a\sec^3 t}{\sec^4 t \cdot \sin t}$$

Thus, $\rho = a\cot t$

13) Find the radius of curvature of astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \pi/4$

Sol^n :⇒

$x = a\cos^3\theta$; $y = a\sin^3\theta$

$\therefore \frac{dx}{d\theta} = -3a\cos^2\theta \cdot \sin\theta$; $\frac{dy}{d\theta} = 3a\sin^2\theta \cdot \cos\theta$

Now, $y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta} = -\tan\theta$$

Further, $y_2 = -\sec^2\theta \cdot \frac{d\theta}{dx}$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta \cdot \sin\theta} = \frac{\sec^4\theta \cdot \operatorname{cosec}\theta}{3a}$$

we have $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2\theta)^{3/2} \cdot 3a}{\sec^4\theta \cdot \operatorname{cosec}\theta}$$

$$= \frac{3a\sec^3\theta}{\sec^4\theta \cdot \operatorname{cosec}\theta} = 3a\cos\theta \cdot \sin\theta$$

Thus at $\theta = \frac{\pi}{4}$

$$\rho = 3a/2$$

Since $\cos \pi/4 = 1/\sqrt{2} = \sin(\pi/4)$

14) Show that the radius of curvature of the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ is 'at'

Sol^n :⇒ $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$;

$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$

$\therefore y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$

Further, $y_2 = \sec^2 t \cdot \frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at}$

we have,

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2 t)^{3/2}}{\sec^3 t} \cdot at$

$= \frac{\sec^3 t \cdot at}{\sec^3 t}$

Thus, $\rho = at$

15> If ρ be the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$, show that ρ^2 varies as $(SP)^3$ where S is the focus of the parabola

Sol^n :⇒ Consider, $y^2 = 4ax$

Diff w.r.to x

$\therefore 2yy_1 = 4a$ or $y_1 = 2a/y$

Further, $y_2 = \frac{-2a}{y^2} \cdot y_1 = \frac{-4a^2}{y^3}$

we have,

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{\{1 + (4a^2/y^2)\}^{3/2}}{-4a^2/y^3}$

$= \frac{y^3\{(y^2+4a^2)/y^2\}^{3/2}}{-4a^2}$

$= \frac{y^3}{-4a^2} \cdot \frac{(y^2+4a^2)^{3/2}}{(y^2)^{3/2}} = \frac{(y^2+4a^2)^{3/2}}{-4a^2}$

$\rho = \frac{(4ax + 4a^2)^{3/2}}{-4a^2}$

$\rho = \frac{(4a)^{3/2}(x+a)^{3/2}}{-4a^2}$

By squaring we have

$\rho^2 = \frac{(4a)^3(x+a)^3}{16a^4}$

$= \frac{64a^3(x+a)^3}{16a^4}$

i.e $\rho^2 = \frac{4}{a}(x+a)^3$ —— ①

The co-ordinates of the focus of the parabola is $S = (a, 0)$ and we have $P = (x, y)$

$\therefore SP = \sqrt{(x-a)^2 + (y-0)^2}$ (by distance formula)

$= \sqrt{x^2 - 2ax + a^2 + y^2}$

$= \sqrt{x^2 - 2ax + a^2 + 4ax}$

$= \sqrt{x^2 + 2ax + a^2}$

$= \sqrt{(x+a)^2}$

$= x + a.$

Hence, $SP = (x+a)$ and using this result in ① we have,

$$\rho^2 = \frac{4}{a}(SP)^3$$

i.e $\rho^2 = \text{const}\cdot(SP)^3$

$\Rightarrow \rho^2$ varies as $(SP)^3$

Thus $\rho^2 \propto (SP)^3$

Expression for the radius of Curvature for a polar curve $r=f(\theta)$

Denoting, $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

the Expression for ρ can be proved in the following form

$$\boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\cdot r_2}}$$

Problems :⇒

1) Show that for the Equiangular Spiral $r=a\cdot e^{\theta\cot\alpha}$ where a and α are constants, ρ/r is a constant

Sol^n :⇒ $r = a\cdot e^{\theta\cdot\cot\alpha}$

$\log r = \log a + \theta\cdot\cot\alpha\cdot\log e$ $\quad \log e = 1$

Diff w.r.to θ. we have

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1\cdot\cot\alpha$$

or $\frac{dr}{d\theta} \Rightarrow r_1 = r\cdot\cot\alpha$

Hence, $\frac{d^2r}{d\theta^2} = r_2 = r_1\cot\alpha = (r\cot\alpha)\cot\alpha = r\cot^2\alpha$

we have, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$$\rho = \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r^2\cot^2\alpha}$$

$$= \frac{(r^2)^{3/2}\cdot(\operatorname{cosec}^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$= \frac{r^3\operatorname{cosec}^3\alpha}{r^2\operatorname{cosec}^2\alpha}$$

Thus $\rho/r = \operatorname{cosec}\alpha = \text{constant}$

2) Show that ρ for the curve $r^n = a^n\cos n\theta$ Varies inversely as r^{n-1}

or

Find the radius of curvature for the curve $r^n = a^n\cos n\theta$.

Sol^n :⇒ $r^n = a^n\cos n\theta$

$n\log r = n\log a + \log\cos n\theta$.

Diff w.r.to θ we have

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n\sin n\theta}{\cos n\theta}$$

or

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$r_1 = -r\tan n\theta$.

Further, $r_2 = \frac{d^2r}{d\theta^2} = -r_1\tan n\theta - nr\sec^2 n\theta$

we have, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$

$\therefore \rho = \dfrac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r(-r_1 \tan n\theta - nr\sec^2 n\theta)}$

$= \dfrac{(r^2)^{3/2}\cdot(\sec^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$

$\rho = \dfrac{r^3\sec^3 n\theta}{r^2(1+\tan^2 n\theta + n\sec^2 n\theta)}$

$= \dfrac{r\sec^3 n\theta}{\sec^2 n\theta(1+n)}$

$= \dfrac{r\sec n\theta}{1+n}$

Hence, $\rho = \dfrac{r}{1+n}\sec n\theta$ But $\dfrac{a^n}{r^n} = \sec n\theta$. (By data)

$\therefore \rho = \dfrac{r}{1+n}\cdot\dfrac{a^n}{r^n}$

$\rho = \left[\dfrac{a^n}{1+n}\right]\cdot\dfrac{1}{r^{n-1}}$

That is, $\rho = \text{const}\cdot\dfrac{1}{r^{n-1}}$

Thus, $\rho \propto \dfrac{1}{r^{n-1}}$

Note : Similar problem.

Find the radius of curvature for the curve $r^n = a^n \sin n\theta$.

$\Rightarrow$ Proceeding on the same lines we can obtain,

$\rho = \dfrac{r}{1+n}\operatorname{cosec} n\theta$ But $\dfrac{a^n}{r^n} = \operatorname{cosec} n\theta$. (By data)

Thus $\rho = \dfrac{a^n}{1+n}\cdot\dfrac{1}{r^{n-1}}$

37) Show that for the curve $r(1-\cos\theta) = 2a$. ρ^2 varies as r^3

Sol^n: $\Rightarrow$ $r(1-\cos\theta) = 2a$

$\Rightarrow \log r + \log(1-\cos\theta) = \log 2a$

Diff w.r.to θ we get

$\dfrac{1}{r}\dfrac{dr}{d\theta} + \dfrac{\sin\theta}{1-\cos\theta} = 0$

or $\dfrac{dr}{d\theta} = \dfrac{-r\sin\theta}{1-\cos\theta}$.

i.e $\dfrac{dr}{d\theta} = \dfrac{-2r\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)}$

$= -r\cot(\theta/2)$

i.e $r_1 = -r\cot(\theta/2)$

Further, $r_2 = -r\cdot\dfrac{-1}{2}\operatorname{cosec}^2(\theta/2) - r_1\cot(\theta/2)$

i.e $r_2 = \dfrac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cot^2(\theta/2)$

we have, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$\therefore \rho = \dfrac{\{r^2 + r^2\cot^2(\theta/2)\}^{3/2}}{r^2 + 2r^2\cot^2(\theta/2) - \dfrac{r^2}{2}\operatorname{cosec}^2(\theta/2) - r^2\cot^2(\theta/2)}$

$= \dfrac{(r^2)^{3/2}\{\operatorname{cosec}^2(\theta/2)\}^{3/2}}{r^2\left\{1+\cot^2\left(\frac{\theta}{2}\right) - \frac{1}{2}\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\right\}}$

$= \dfrac{r\operatorname{cosec}^3(\theta/2)}{1/2\operatorname{cosec}^2(\theta/2)}$

$\rho = 2r\operatorname{cosec}(\theta/2)$ —— (1)

But, $r(1-\cos\theta) = 2a$ By data

i.e $r \cdot 2\sin^2(\theta/2) = 2a$ or $\sin^2(\theta/2) = a/r$

$\therefore \operatorname{cosec}(\theta/2) = \sqrt{r/a}$ and hence ① becomes

$\rho = 2r \cdot \sqrt{r/a} = 2r^{3/2}/\sqrt{a}$

Hence

$\rho^2 = 4r^3/a = (4/a) \cdot r^3$

Thus, $\rho^2 \propto r^3$

4> Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole

Sol^n:⇒ $r = a\sin n\theta$

$\therefore r_1 = an\cos n\theta$

$r_2 = -an^2\sin n\theta$

At the pole we have, $\theta = 0$

when $\theta = 0$, $r = 0$, $r_1 = an$, $r_2 = 0$

we have, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r \cdot r_2}$

$\therefore \rho = \dfrac{(a^2n^2)^{3/2}}{2a^2n^2} = \dfrac{a^3n^3}{2a^2n^2} = \dfrac{an}{2}$

Thus $\rho = \dfrac{an}{2}$ at the pole.

5> Show that at the point where the curve $r = a\theta$ intersects the curve $r = a/\theta$ their curvatures are in the ratio 3:1

Sol^n:- Equating the RHS of the given Equations

$r = a\theta$ and $r = a/\theta$ we have,

$a\theta = \dfrac{a}{\theta}$ or $\theta^2 = 1$, $\therefore \theta = \pm 1$

Now, $r = a\theta$, gives $r_1 = a$, $r_2 = 0$

At $\theta = \pm 1$, $r = a$, $r_1 = a$, $r_2 = 0$ —— ①

Also, $r = a/\theta$ gives $r_1 = -a/\theta^2$

$r_2 = 2a/\theta^3$

At $\theta = \pm 1$, $r = a$, $r_1 = -a$, $r_2 = 2a$ —— ②

we have, $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r \cdot r_2}$

From ①, $\rho = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2} = \dfrac{(2a^2)^{3/2}}{3a^2} = \dfrac{2\sqrt{2}a}{3}$ —— ③

From ②, $\rho = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2-2a^2} = \dfrac{(2a^2)^{3/2}}{a^2} = \dfrac{2\sqrt{2}a}{1}$ —— ④

Hence we have from ③ and ④ the ratio of the corresponding curvatures is given by

$\dfrac{3/2\sqrt{2} \cdot a}{1/2\sqrt{2}a} = \dfrac{3}{1}$, since curvature $K = \dfrac{1}{\rho}$

Thus the curvatures are in the ratio 3:1

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6> a> Show that for the curve $r = a(1+\cos\theta)$, ρ^2/r is a constant

b> If ρ_1 and ρ_2 be the radii of curvatures at the extremities of the polar chord of the cardioide, S.T $\rho_1^2 + \rho_2^2 = 16a^2/9$

Sol^n:⇒

a> $r = a(1+\cos\theta)$

⇒ $\log r = \log a + \log(1+\cos\theta)$

Diff w.r.to θ we have

$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-\sin\theta}{1+\cos\theta} = \dfrac{-2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2}$

$\therefore r_1 = -r\tan\theta/2$

Further, $r_2 = -\frac{r}{2}\sec^2(\theta/2) - r_1\tan(\theta/2)$

i.e $r_2 = -\frac{r}{2}\sec^2(\theta/2) + r\tan^2(\theta/2)$

We have, $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\cdot r_2}$

$\therefore \rho = \frac{\{r^2+r^2\tan^2(\theta/2)\}^{3/2}}{r^2+2r^2\tan^2(\theta/2)+\frac{r^2}{2}\sec(\theta/2)-r^2\tan^2(\theta/2)}$

$= \frac{r^3\{\sec^2(\theta/2)\}^{3/2}}{r^2\{1+\tan^2(\frac{\theta}{2})+\frac{1}{2}\sec^2(\frac{\theta}{2})\}}$

$= \frac{r\sec^3(\theta/2)}{3/2\cdot\sec^2(\theta/2)} = \frac{2r}{3}\sec(\theta/2)$

$\rho = \frac{2r}{3}\sec(\theta/2)$ —— ①

But, $r = a(1+\cos\theta) = a\cdot 2\cos^2(\theta/2)$

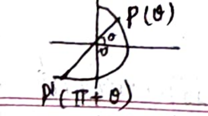
$\therefore \sec^2(\theta/2) = \frac{2a}{r}$ or $\sec(\theta/2) = \frac{\sqrt{2a}}{\sqrt{r}}$

Hence ① becomes $\rho = \frac{2r}{3}\cdot\frac{\sqrt{2a}}{\sqrt{r}}$

i.e $\rho = \frac{2}{3}\sqrt{2ar}$

$\therefore \rho^2 = \frac{4}{9}(2ar)$ or $\frac{\rho^2}{r} = \frac{8a}{9}$ = constant

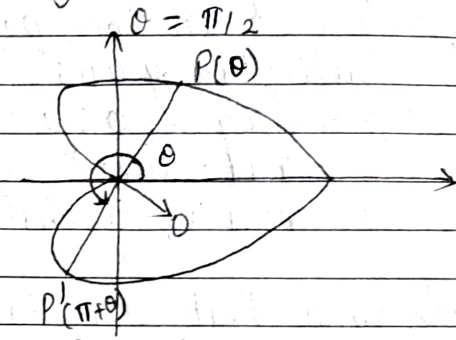
Thus, $\frac{\rho^2}{r}$ is a constant



b) Let POP' be the polar chord (chord passing through the pole) of the cardioide $r = a(1+\cos\theta)$. Let ρ_1 and ρ_2 be the radii of curvatures at the point P and P' corresponding to the vectorial angles θ and $(\pi+\theta)$ respectively. We have already obtained

$\rho_1 = \frac{2r}{3}\sec(\theta/2)$

(first part of this problem)



$\rho_1^2 = \frac{4r^2}{9}\sec^2(\theta/2)$

But, $r = a(1+\cos\theta) = 2a\cos^2(\theta/2)$

$\therefore r^2 = 4a^2\cos^4(\theta/2)$

Hence, $\rho_1^2 = \frac{4}{9}\cdot 4a^2\cos^4(\theta/2)\cdot\sec^2(\theta/2)$

i.e $\rho_1^2 = \frac{16a^2}{9}\cos^2(\theta/2)$ —— ②

Now changing θ to $(\pi+\theta)$, we have from ②

$\rho_2^2 = \frac{16a^2}{9}\cos^2\left(\frac{\pi+\theta}{2}\right)$

$= \frac{16a^2}{9}\cdot\cos^2\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

i.e $\rho_2^2 = \frac{16a^2}{9}\sin^2(\theta/2)$ —— ③

We shall add ② & ③

Thus $\rho_1^2+\rho_2^2 = \frac{16a^2}{9}$ = constant.

Denoting $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$

and $x'' = \frac{d^2x}{dt^2}$, $y'' = \frac{d^2y}{dt^2}$

The radius of curvature is

$$\rho = \frac{\{(x')^2 + (y')^2\}^{3/2}}{x'y'' - y'.x''}$$

Expression for the radius of Curvature for a pedal curve.

$$\rho = r\frac{dr}{dp}$$

ϕ

$p = r\sin\phi$

$p = r$

$\frac{dp}{dr}$

$\frac{dr}{dp}$

$r.\frac{dr}{dp}$

$$\rho = r.\frac{dr}{dp}$$

1) prove that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of curvature is equal to $\frac{a^2b^2}{p^3}$ where p is the length of perpendicular from the centre of ellipse upon the tangent at (x,y). Hence deduce that ρ at the end of the major axis is equal to the semi latus rectum.

Sol^n:⇒ The parametric equ^{n}s of the ellipse are $x=a\cos\theta$, $y=b\sin\theta$ and we prefer to apply the parametric formula for finding ρ.

$$\rho=\frac{\{(x')^2+(y')^2\}^{3/2}}{x'y''-y'x''}$$ for a parametric curve

$x'=\frac{dx}{d\theta}=-a\sin\theta$, $y'=\frac{dy}{d\theta}=b\cos\theta$

$x''=\frac{d^2x}{d\theta^2}=-a\cos\theta$, $y''=\frac{d^2y}{d\theta^2}=-b\sin\theta$

$$\therefore \rho=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab(\sin^2\theta+\cos^2\theta)}$$

$$\rho=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}$$

Further the Eq^n of the tangent to the ellipse at $P(a\cos\theta, b\sin\theta)$ is given by $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. Also the length of the perpendicular from the point (x_1,y_1) upon a straight line $Ax+By+c=0$ is given by the formula

$$p=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

Hence the length of the perpendicular from the centre $O=(0,0)$ of the ellipse upon the tangent $(\cos\theta/a)x+(\sin\theta/b)y-1=0$ is given by

$$p=\frac{|0+0-1|}{\sqrt{\cos^2\theta/a^2+\sin^2\theta/b^2}}$$

$$=\frac{1}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta/a^2b^2}}$$

$$p=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

$$p^3=\frac{a^3b^3}{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}$$

$$\therefore (a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}=\frac{a^3b^3}{p^3}$$

$\rho=\frac{[a^3b^3]^{3/2}}{p^3\, ab}$ Further

$\rho=$ $\rho=\frac{a^3b^3/p^3}{ab}$

$$\rho=\frac{a^2b^2}{p^3}$$

Further at the end of major axis we have $(x,y)=(\pm a,0)$

$\therefore a\cos\theta=\pm a \Rightarrow \cos\theta=\pm1$ or $\cos^2\theta=1$, $\therefore \sin^2\theta=0$

Hence $p=\frac{ab}{\sqrt{0+b^2}}$ or $p=a$

Thus $\rho=\frac{a^2b^2}{a^3}=\frac{b^2}{a}$ being a length of semi latus rectum

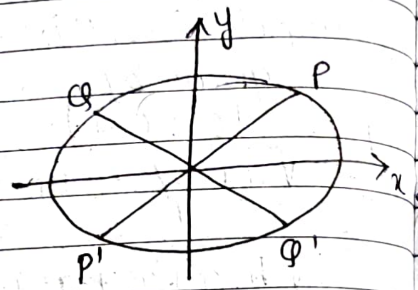
This proves both the desired results.

27. If ρ_1 and ρ_2 be the radii of curvature at the extrimities of the two conjugate diameters of the ellipse, show that

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{(a^2+b^2)}{(ab)^{2/3}}$$

Sol^n ⇒

Let PCP' and QCQ' be the two conjugate diameters of the ellipse



Noting that $x = a\cos\theta$ by $y = b\sin\theta$ represents the parametric equations of the ellipse and recollecting a property of the conjugate diameters with reference to the eccentric angle θ, we can write.

$P = (a\cos\theta, b\sin\theta)$ and

$Q = [a\cos(\pi/2+\theta), b\sin(\pi/2+\theta)]$

Let ρ_1 be the radius of curvature at P and ρ_2 be the radius of curvature at Q

Hence,

$$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} \quad \text{—— (1)}$$

changing θ to $\pi/2 + \theta$ we have

$\sin(\pi/2+\theta) = \cos\theta$ and

$\cos(\pi/2+\theta) = -\sin\theta$. Thus we have from (1)

$$\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$$

i.e $\cancel{\frac{d\phi}{d\theta}} = \frac{r_1^2 - r\cdot r_2}{r_1^2[1+(r^2/r_1^2)]}$

$$= \frac{r_1^2 - r\cdot r_2}{r_1^2 + r^2}$$

Hence $1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - r\cdot r_2}{r^2 + r_1^2}$

$$= \frac{r^2 + r_1^2 + r_1^2 - r\cdot r_2}{r^2 + r_1^2}$$

i.e $1+d$

$$\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab} \quad \text{—— (2)}$$

Adding eq^n (1) and (2).

$$\rho_1 + \rho_2 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} + \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$$

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta) + (a^2\cos^2\theta + b^2\sin^2\theta)}{ab^{2/3}}$$

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2(\sin^2\theta + \cos^2\theta) + b^2(\cos^2\theta + \sin^2\theta)}{ab^{2/3}}$$

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{(a^2+b^2)}{(ab)^{2/3}}$$

3) Show that for the ellipse in the pedal form $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$ the radius of Curvature at the point (p, r) is a^2b^2/p^3

Sol^n :⇒ Consider $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

Diff w.r.to p we get

$$\frac{-2}{p^3} = \frac{-2r}{a^2b^2} \cdot \frac{dr}{dp}$$

$$\therefore \frac{dr}{dp} = \frac{-\cancel{2}a^2b^2}{-\cancel{2}p^3 r}$$

$$\frac{dr}{dp} = \frac{a^2b^2}{p^3 \cdot r}$$

we have $\rho = r \cdot \frac{dr}{dp}$

and hence $\rho = \cancel{r} \cdot \frac{a^2b^2}{p^3 \cdot \cancel{r}} = \frac{a^2b^2}{p^3}$

Thus $\rho = \frac{a^2b^2}{p^3}$.

4) Find the radius of Curvature of the Curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it.

Sol^n :⇒ Diff w.r.to r.

$$\frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{2r}{2\sqrt{r^2-a^2}} - \left\{ \frac{-1}{\sqrt{1-(a/r)^2}} \cdot \frac{-a}{r^2} \right\}$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{r}{\sqrt{r^2-a^2}} \cdot \frac{a}{r^2}$$

$$= \frac{1}{\sqrt{r^2-a^2}} \left(\frac{r}{a} - \frac{a}{r} \right) = \frac{r^2-a^2}{\sqrt{r^2-a^2} \cdot ar}$$

i.e $\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$ —— ①

$$r\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{a}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

$$\cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

consider $p = r\sin\phi$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} (1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \frac{a^2}{(\sqrt{r^2-a^2})^2} \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \frac{a^2}{r^2-a^2} \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \left[\frac{r^2 - \cancel{a^2} + \cancel{a^2}}{r^2-a^2} \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\cancel{r^2}} \left[\frac{\cancel{r^2}}{r^2-a^2} \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2-a^2}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{\sqrt{r^2-a^2}}$$

$\Rightarrow p = \sqrt{r^2-a^2}$ is the pedal Eq^n of the curve

Diff w.r to p we get

$$1 = \frac{2r}{2\sqrt{r^2-a^2}} \frac{dr}{dp}$$

$$\sqrt{r^2-a^2} = r.\frac{dr}{dp} = \rho$$

$$\rho = \sqrt{r^2-a^2}$$