Module - 5

Numerical method for initial value problem

Consider a differential equation of 1st order and first degree in the form $\frac{dy}{dx} = f(x,y) \rightarrow (1)$

with the initial condition $y(x_0) = y_0$ - (2) [that means when $x = x_0$, $y = y_0$]

The problem of solving equation (1) subject to the condition eq (2) is called initial value problem [IVP]

Taylor's series expansion / Method

Consider the initial value problem

$\frac{dy}{dx} = f(x,y)$ and $y(x_0) = y_0$ we have taylor series expansion of $y(x)$ about the point x_0

in the form

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

here $y'(x_0)$, $y''(x_0)$, $y'''(x_0)$ .... denotes the value of derivative $\frac{dy}{dx}$, $\frac{d^2y}{dx}$, $\frac{d^3y}{dx^3}$ ..... at x_0

solve $\frac{dy}{dx} = x^2 y - 1$ with $y(0) = 1$ using taylor series and find $y(0.1)$ by considering upto 4th degree term

Given

$f(x,y) = y'(x) = x^2 y - 1$

$y(0) = 1 \quad x_0 = 0, y_0 = 1$

taylor series expansion

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2 y''(x_0)}{2!} + \frac{(x-x_0)^3 y'''(x_0)}{3!}$$

$$+ \frac{(x-x_0)^4 y''''(x_0)}{4!}$$

$y'(x) = x^2 y - 1$

$= 0 \times 1 - 1$

$y'(x) = -1$

$y''(x) = 2xy$

$= 2 \times 0 \times 1$

$y''(x) = 0$

$y'''(x) = 2[xy' + y]$

$= 2xy' + 2y$

$= 2 \times (0)(-1) + 2(1)$

$y'''(x) = 2$

$y''''(x) = 2[xy'' + y'] + 2y'$

$= 2xy'' + 2y' + 2y'$

$= 2(0)(0) + 2(-1) + 2(-1)$

$y'(x) = x^2 y - 1$

$= 0 \times 1 - 1$

$y'(x) = -1$

$y''(x) = x^2 y' + 2xy$

$= 0 \times -1 + 2(0)1$

$= 0$

$y'''(x) = x^2 y'' + 2xy' + 2[xy' + y]$

$= x^2 y'' + 2xy' + 2xy' + 2y$

$= x^2 y'' + 4xy' + 2y$

$= (0)^2 (0) + 4 \times 0 \times -1 + 2(1)$

$= 2$

$y''''(x) = x^2 y''' + 2xy'' + 2xy'' + 2y' + 2xy'' + 2y' + 2y'$

$= x^2 y''' + 2xy'' + 4xy'' + 6y'$

$y''''(x) = -6$

$$y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2 \times 0}{2 \times 1} + \frac{(x-0)^3 \times 2}{6} + \frac{(x-0)^4 \times -6}{24}$$

$$= 1 - x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$

$$\boxed{y(0.1) = 0.90033}$$

use Taylor's series method to find y at $x = 0.1$ and at $x = 0.2$ given $y' = x^2 + y^2$ with $y(0) = 1$ considering upto 4th degree term

Given

$f(x,y) = y'(x) = x^2 + y^2$

$y(0) = 1 \quad x_0 = 0 \quad y_0 = 1$

$y'(x) = x^2 + y^2$

$= 0^2 + 1^2$

$= 1$

$y''(x) = 2x + 2yy'$

$= 2(0) + 2 \times 1 \times 1$

$= 2$

$y'''(x) = 2 + 2[yy'' + y'y']$

$= 2 + 2yy'' + 2y'y'$

$= 2 + 2(1) \times 2 + 2(1 \times 1)$

$= 2 + 4 + 2$

$= 8$

$$y''''(x) = 2[yy''' + y''y'] + 2[y'y'' + y'y'']$$

$$= 2yy''' + 2y''y' + 2y'y'' + 2y'y'']$$

$$= 2yy''' + 6y'y''$$

$$= 2(1)\times 8 + 6(1)\times 2$$

$$= 16 + 12$$

$$= 28$$

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y''''(x_0)$$

$$= 1 + (x-0)\times 1 + \frac{(x-0)^2}{2}\times 2 + \frac{(x-0)^3}{6}\times 7 + \frac{(x-0)^4}{24}\times 28$$

$$y(0.1) = 1 + x + x^2 + 8\frac{x^3}{6} + \frac{x^4}{24}\times 28$$

$$= 1 + (0.1) + (0.1)^2 + 8\frac{(0.1)^3}{6} + \frac{(0.1)^4}{24}\times 28$$

$$y(0.1) = 1.1114$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{8(0.2)^3}{6} + \frac{(0.2)^4}{24}\times 28$$

$$= 1.2525$$

Employ taylor series method to obtain approximate value of y at $x = 0.1$ and $x = 0.2$ for differential equation $\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 0$ considering upto 4 degree term

Given

$$f(x,y) = y'(x) = 2y + 3e^x$$

$y(0) = 0$ $\quad x_0 = 0 \quad y_0 = 0$

$$y'(x) = 2y + 3e^x$$

$$= 2(0) + 3e^0$$

$$= 3$$

$$y''(x) = 2y' + 3 \times e^x$$

$$= 2 \times 3 + 3 \times e^0$$

$$= 9$$

$$y'''(x) = \cancel{2[y y'' + y'y'] + 3xe^x} \quad 2[y''] + 3e^x$$

$$2[9] + 3e^0$$

$$\cancel{= 2yy'' + 2y'y' + 3xe^x}$$

$$\cancel{= 2 \times 0 \times 3 + 2 \times 3 \times 3 + 3 \times e^0} \quad = \cancel{6+3}\ 18+3$$

$$\cancel{= 18 + 3} \quad = 21$$

$$= 21$$

$\cancel{y''''(x) = 2[yy''' + y''y'] + 2[y'y'' + y'y'']} +$

$3 \times e^{x}$

$= \cancel{2yy''' + 2y''y' + 2y'y'' + } 2y'$

$y''''(x) = 2y''' + 3e^{x}$

$= 2[21] + 3e^{0}$

$= 45$

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_{0})$$

$$+ \frac{(x - x_0)^3}{3!} y'''(x_0) + \frac{(x - x_0)^4}{4!} y''''(x_0)$$

$$= 0 + (x - 0) \times 3 + \frac{(x-0)^2}{2} \times 9 +$$

$$\frac{(x-0)^3}{6} \times 21 + \frac{(x-0)^4}{24} \times 45$$

$$y(x) = 3x + \frac{9x^2}{2} + \frac{21x^3}{6} + \frac{45x^4}{24}$$

$$y(0.1) = 3(0.1) + \frac{9(0.1)^2}{2} + \frac{21(0.1)^3}{6} + \frac{45(0.1)^4}{24}$$

$= 0.3487$

$$y(0.2) = 3(0.2) + \frac{9(0.2)^2}{2} + \frac{21(0.2)^3}{6} + \frac{45(0.2)^4}{24}$$

$y(0.2) = 0.811$

use taylor series method to solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking step size $h = 0.1$ Given that $y = 10$ at $x = 0$ initially considering upto 4th degree term

Given

$$f(x, y) = y'(x) = x^2 + y$$

$$y_0 = 10 \qquad x_0 = 0$$

since the step size is 0.1 the problem has to be solved in two stages

First we have to find $y(0.1)$ and using this as initial condition we have to compute 0.2

$$y'(x) = x^2 + y$$
$$= 0 + 10$$
$$= 10$$

$$y''(x) = 2x + y'$$
$$= 2(0) + 10$$
$$= 10$$

$$y'''(x) = 2 + y''$$
$$= 2 + 10$$
$$= 12$$

$$y''''(x) = y'''$$
$$= 12$$

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0)$$
$$+ \frac{(x - x_0)^3}{3!} y'''(x_0) + \frac{(x - x_0)^3}{4!} y''''(x_0)$$
$$= 10 + (x - 0)\, 10 + \frac{(x - 0)^2}{2} \times 10 +$$
$$\frac{(x - 0)^3}{6} \times 12 + \frac{(x - 0)^4}{24} \times 12$$

$$10 + 10x + \frac{10x^2}{2} + \frac{12x^3}{6} + \frac{x^4 12}{24}$$

$$= 10 + 10(0.1)$$

$$y(0.1) = 11.052$$

II stage

$$x_0 = 0.1 \quad y_0 = 11.052$$

$$y' = x^2 + y$$

$$y' = (0.1)^2 + 11.052$$

$$y' = 11.062$$

$$y''(x_0) = 2x + y'$$

$$= 2(0.1) + 11.062$$

$$= 11.262$$

$$y'''(x_0) = 2 + y''$$

$$= 2 + 11.262$$

$$= 13.262$$

$$y''''(x_0) = y'''$$

$$= 13.262$$

$$y(x) = y(x_0) + (x-x_0)\frac{y'(x_0)}{1!} + \frac{(x-x_0)^2 y''(x_0)}{2!} + \frac{(x-x_0)^3 y'''(x_0)}{3!} + \frac{(x-x_0)^4 y''''(x_0)}{4!}$$

$$= 11.052 + \frac{(0.2-0.1)}{1} \times 11.062 + \frac{(0.1)^2 \times 11.262}{2} + \frac{(0.1)^3 \times 13.262}{6} + \frac{(0.1)^4}{24} \times 13.262$$

$$= 12.21677$$

$$= 12.2168$$

Solve $\frac{dy}{dx} = x - y^2$ with $y(0) = 1$ using taylor series method and find the value of y at $x = 0.1$

H.W by considering unto 4 degree terms.

Given

$f(x,y) = y'(x) = x - y^2$

$y_0 = 1 \quad x_0 = 0$

$y_1' = x - y^2$

$= 0 - 1^2$

$= -1$

$y''(x) = 1 - 2yy'$

$= 1 - 2(1)(-1)$

$= 1 + 2$

$= 3$

$$y'''(x) = -2[yy'' + y'y']$$

$$= -2yy'' - 2y'y'$$

$$= -2(1)3 - 2(-1)(-1)$$

$$= -6 - 2$$

$$= -8$$

$$y''''(x) = -2[y'''y + y''y'] - 2[y'y'' + y'y'']$$

$$= -2y'''y - 2y''y' - 2y'y'' - 2y'y''$$

$$= -2y'''y - 6y''y'$$

$$= -2(-8)(-1) - 6(3)(-1)$$

$$= -16 + 18$$

$$= 2$$

$$y(x) = y(x_0) + (x-x_0)y'(x) + \frac{(x-x_0)^2 y''(x)}{2!} + \frac{(x-x_0)^3 y'''(x)}{3!} + \frac{(x-x_0)^4 y^{iv}(x)}{4!}$$

$$= 1 + (x-0)-1 + \frac{(x-0)^2 3}{2} + \frac{(x-0)^3 -8}{6} + \frac{(x-0)^4 2}{24}$$

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{8x^3}{6} + \frac{x^4}{12}$$

$$y(0.1) = 1 - (0.1) + \frac{3(0.1)^2}{2} - \frac{8(0.1)^3}{6} + \frac{(0.1)^4}{12} = 0.91367$$

solve $\frac{dy}{dx} = e^x - y^2$ with $y(1) = 1$ using taylor series method & find $y(0.1)$ by considering upto 4th degree term

Given

$F(x,y) = y'(x) = e^x - y^2$

$x_0 = 1 \qquad y_0 = 1$

$$y'(x) = e^x - y^2$$
$$= e^1 - (1)^2$$
$$= 1.71838$$

$$y''(x) = e^x - 2yy'$$
$$= e^1 - 2(1)(1.71838)$$
$$= -0.7183$$

$$y'''(x) = e^x - 2[y''y + y'y']$$
$$= e^x - 2y''y - 2y'y'$$
$$= e^1 - 2(-0.7183)(1) - 2(1.7183 \times 1.7183)$$
$$= \cancel{-0.7183}$$
$$= -1.7502$$

$$y''''(x) = e^x - 2[y''y' + y'''y] - 2[y'y'' + y''y']$$
$$= e^x - 2y''y' - 2y'''y - 2y'y'' - 2y''y'$$
$$= e^1 - 6y''y' - 2y'''y$$

$= e^1 - 6(-0.7183) \times 1.7183 - 2(-1.7502 \times 1)$

$= 13.624$

$$y(0.1) = y(x_0) + (x - x_0) y'(x) + \frac{(x-x_0)^2 y''(x)}{2!} + \frac{(x-x_0)^3 y'''(x)}{3!} + \frac{(x-x_0)^4 y^{iv}(x)}{4!}$$

$$= 1 + (-0.9) \times 1.7183 + \frac{(-0.9)^2}{2}(-0.7183) + \frac{(-0.9)^3}{6} \times (-1.7502) + \frac{(-0.9)^4}{24} \times 13.624$$

$y(0.1) = -0.2523$

Modified Euler's method

consider the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

The initial approximation for y_1 at $x_1 = x_0 + h$ is found using Euler's formula given by

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

The other approximations are found using modified Eulers formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Note: Euler's formula and modified Euler's formula jointly called as Euler's predictor and corrector formula

Problem

1. use modified Euler's method to solve the initial value problem

$y' = x^2 + y^2$, $y(0) = 0$ find $y(0.1)$

Given

$y' = f(x,y) = x^2 + y^2$

$x_0 = 0 \qquad y_0 = 0 \qquad x_1 = x_0 + h$

$= 0.1$

Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 0 + 0.1(x^2 + y^2)$$

$$= 0 + 0.1(0^2 + 0^2)$$

$$= 0$$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 0 + \frac{0.1}{2}\left[0 + (0.1)^2 + 0^2\right]$$

$$= 0.0005$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1\, y_1^{(1)}) \right]$$

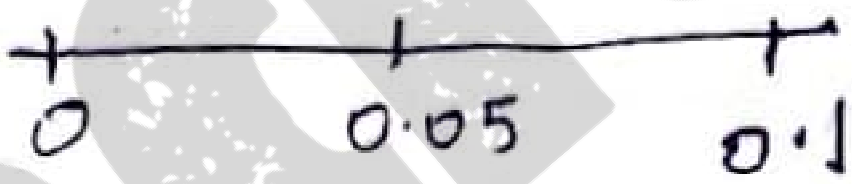
$$= 0 + \frac{0.1}{2}\left[0 + (0.1)^2 + (0.0005)^2 \right]$$

$$y_1^{(2)} = 0.0005$$

$$\therefore y(0.1) = 0.0005$$

determine the value of y when $x = 0.1$ Given that $y(0) = 1$ and $y' = x^2 + y$ using modified Euler's formula take $h = 0.05$

Given



$$f(x,y) = y' = x^2 + y$$

$x_0 = 0 \qquad y_0 = 1 \qquad x_1 = x_0 + h$

$$= 0 + 0.05$$

$$= 0.05$$

Eulers formula

$$y_1^{(0)} = y_0 + h f(x_0 y_0)$$

$$= 1 + 0.05\,(0^2 + 1)$$

$$= 1.05$$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1, y_1^{(0)}) \right]$$

$$= 1 + \frac{0.05}{2}\left[1 + (0.05)^2 + 1.05 \right]$$

$$= 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1, y_1^{(1)}) \right]$$

$$= 1 + \frac{0.05}{2}\left[1 + (0.05)^2 + 1.0513 \right]$$

$$= 1.0513$$

$$\boxed{y(0.05) = 1.0513}$$

By stage 2

$x_0 = 0.05 \qquad y_0 = 1.0513 \qquad h = 0.05$

$$x_1 = x_0 + h = 0.05 + 0.05 = 0.1$$

Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0 y_0)$$

$$= 1.0513 + 0.05\left((0.05)^2 + (1.0513)^2 \right)$$

$$= 1.10\cancel{39}4$$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(0)}) \right]$$

$$= 1.0513 + \frac{0.05}{2}\left[(0.05)^2 + 1.0513 + (0.1)^2 + 1.104 \right]$$

$$= 1.1055$$

$$y_2^{(2)} = 1.0513 + \frac{0.05}{2}\left[(0.05)^2 + 1.0513 + (0.1)^2 + 1.1055 \right]$$

$$= 1.1055$$

$$y(0.1) = 1.1055 \ //$$

Using Euler's predictor & corrector formula compute $y(2.5)$ when $h = 0.25$ from $\frac{dy}{dx} = \frac{x+y}{x}$ $y(2) = 2$

Given

$$f(x, y) = y'(x) = \frac{x+y}{x}$$

2 2.25 2.5

$x_0 = 2$ $y_0 = 2$ $h = 0.25$

To find $y(x_1) = y(2.25)$

Euler's formula / predictor

$$y_1^{(0)} = y_0 + h f(x_0 y_0)$$

$$= 2 + 0.25\left(\frac{2+2}{2}\right)$$

$$= 2.5$$

By modified Euler's formula / corrector

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 2 + \frac{0.25}{2}\left[2 + \left(\frac{2.25 + 2.5}{2.25}\right)\right]$$

$$= 2.5139$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.25}{2}\left[2 + \frac{2.25 + 2.5139}{2.25}\right]$$

$$= 2.51847$$

$$y_1^{(3)} = 2 + \frac{0.25}{2}\left[2 + \frac{2.25 + 2.5147}{2.25}\right]$$

$$= 2.5147$$

$y(2.25) = 2.5147$

Stag 2

$x_0 = 2.25$ $\quad y_0 = 2.5147$ $\quad h = 0.25$

$x_1 = 2.25 + 0.25$

$= 2.50$

Eulers formula

$$y_1^{(0)} = y_0 + h\, f(x_0 y_0)$$

$$= 2.5147 + 0.25(2.1176)$$

$$= 3.0441$$

modified Eulers formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 2.5147 + \frac{0.25}{2}\left[2.1176 + \left(\frac{2.50 + 3.0441}{2.50}\right)\right]$$

$$= 3.0566$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0\, y_0) + f(x_1\, y_1^{(1)})\right]$$

$$= 2.5147 + \frac{0.25}{2}\left[2.1176 + \left(\frac{2.5 + 3.0566}{2.5}\right)\right]$$

$$= 3.0572$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0\, y_0) + f(x_1,\, y_1^{(2)})\right]$$

$$= 2.5147 + \frac{0.25}{2}\left[2.1176 + \left(\frac{2.5 + 3.0572}{2.5}\right)\right]$$

$$= 3.0572$$

$$\therefore y(2.5) = 3.0572$$

Given $y' = \sqrt{xy} = 2$ $y(1) = 1$, find $y(1.25)$ using modified euler's method take $h = 0.25$

$$x_1 = x_0 + h$$
$$= 1.25$$

$$f' = 2 + \sqrt{xy}$$

$$y'(0) = y_0 + h f(x_0\, y_0)$$
$$= 1 + 0.25\,(2 + \sqrt{1 \times 1})$$
$$= 1 + 0.25\,(3)$$
$$= 1.75$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.25}{2}\left[3 + 2 + \sqrt{1.25 \times 1.75}\right]$$

$$= 1.8098$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.25}{2}\left[3 + 2 + \sqrt{1.25 \times 1.8098}\right]$$

$$= \cancel{1.8092}\ 1.8130$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.25}{2}\left(3 + 2 + \sqrt{1.25 \times 1.8130}\right)$$

$$= 1.8132$$

$$y_1^{(4)} = 1 + \frac{0.25}{2}\left(3 + 2 + \sqrt{1.25 \times 1.8132}\right)$$

$$= 1.8132$$

$$y(1.25) = 1.8132$$

Given $\frac{dy}{dx} = 1 + \frac{y}{x}$ $y = 2$ at $x = 1$ find the approximate value of y at $x = 1.4$ by modified Euler's method by taking $h = 0.2$

$\cancel{f' = 1 + \frac{y}{x}}$ $f' = 1 + \frac{y}{x}$

1 1.2 1.4

$\cancel{y'(0) = \phi + \frac{0}{0}}$ $x_0 = 1$ $y_0 = 2$

$x_1 = x_0 + h$

$x_1 = 1 + 0.2$

$= 1.2$

$$y_1^{(0)} = y_0 + h\, f(x_0 y_0)$$

$$= 2 + 0.2\left(1 + \frac{2}{1}\right)$$

$$= 2 + 0.2(3)$$

$$= 2 + 0.6$$

$$= 2.6$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(0)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + \left(1 + \frac{2.6}{1.2}\right)\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.6}{1.2}\right]$$

$$= 2.6167$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(1)})\right]$$

$$= 2.2 + \frac{0.2}{2}\left[2.8333 + 2.9871\right]$$

$$= 2.7820$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(1)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + 1 + \frac{2.6167}{1.2}\right]$$

$$= \cancel{2.2} \; 2.6181$$

$$y_2^{(3)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(2)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + 1 + \frac{2.6181}{1.2}\right]$$

$$= 2.6181$$

stage 2

$x_0 = 1.2 \quad y_0 = 2.2$ $\qquad$ $x_1 = x_0 + h$ $\qquad h = 0.2$

$x_1 = 1.2 + 0.2$

$x_1 = 1.4$

Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0 y_0)$$

$$= 2.2 + 0.2(2.8333)$$

$$= 2.7667$$

$1 + \frac{2.7667}{1.4}$

modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(0)})\right]$$

$$= 2.2 + \frac{0.2}{2}\left[2.8333 + 2.9762\right]$$

$$= 2.7809$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 y_0) + f(x_1 y_1^{(1)})\right]$$

$1 + \frac{2.7809}{1.4}$

$$= 2.2 + \frac{0.2}{2}\left[2.8333 + 2.9863\right]$$

$$= 2.7820$$

$\rightarrow ?$

Runge - kutta method of fourth order

consider a initial value problem $\frac{dy}{dx} = f(x,y)$

we need to find the value of y at $x = x_1 = x_0 + h$,

compute four quantities, namely k_1, k_2, k_3, k_4

where $k_1 = h\, f(x_0\, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right)$$

the value of y at $x = x_1 = x_0 + h$ is given by

$$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Apply Runge kutta method of order four and find an approximate value at y at $x = 0.1$

Given $\frac{dy}{dx} = 3e^x + 2y$ with $y(0) = 0$ & $h = 0.1$

here $f(x,y) = 3e^x + 2y$

$3 \times e^{0.05} + 2(0.15)$

$x_0 = 0 \quad y_0 = 0 \quad x_1 = x_0 + h = 0.1$

$f(x_0\, y_0) = 3 \quad h = 0.1$

$k_1 = h\, f(x_0\, y_0) = 0.1 \times 3 = 0.3$

$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 0.15)$

$= 0.1[3 \times e^{0.05} + 2 \times 0.15] = 0.3454$

$k_3 = h\, f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = 0.1 f(0.05, 0.1727)$

$= 0.3499$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.1 \times f(0.1, 0.3499)$

$= 0.4015$

By runge kutta fourth order method we have

$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0.3 + 2 \times 0.3454 + 2 \times 0.3499 + 0.4015)$

$y(0.1) = 0.3487$

By runge kutta method of order 4 solve $y' = 3x + \frac{y}{2}$ with $y(0) = 1$ & hence find $y(0.1)$ by taking $h = 0.1$

stage 1

$f(x, y) = 3x + \frac{y}{2}$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.1 \quad x_1 = x_0 + h = 0.1$

$f(x_0 y_0) = 0.5$

$k_1 = h f(x_0 y_0) = 0.1 \times 0.5 = 0.05$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times f\left(0 + \frac{0.1}{2}, 1 + 0.025\right)$

$= 0.1 \times f(0.05, 1.025)$

$= 0.0662$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\, f(0.05, 1.0331)$

$= 0.0667$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1\, f(0.1, 1.0667)$

$= 0.0833$

By runge kutta fourth order method we have

$y_1 = f(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.1) = 1 + \frac{1}{6}(0.05 + 2 \times 0.0662 +$

$2 \times 0.0667 + 0.0833)$

$= 1.0665$

Stage

$x_0 = 0.1 \quad y_0 = 1.0665 \quad x_1 = x_0 + h$

$= 0.1 + 0.1$

$= 0.2$

$f(x,y) = 3x + \frac{y}{2}$

$f(x_0 y_0) = 3 \times 0.1 + \frac{1.0665}{2}$

$= 0.8332$

$k_1 = hf(x_0 y_0) = 0.1 \times 0.8332 = ~~0.1666~~\ 0.0833$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.1 f(0.15, 1.1081)$

$$= 0.1\left(3\times 0.15 + \frac{1.1081}{2}\right)$$

$$= 0.1004$$

$$k_3 = hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right]$$

$$= 0.1\left[0.15,\ 1.1167\right]$$

$$= 0.1\left[3\times 0.15 + \frac{1.1167}{2}\right]$$

$$= 0.1008$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

$$= 0.1 f(0.2,\ 1.1673)$$

$$= 0.1\left(3\times 0.2,\ \frac{1.1673}{2}\right)$$

$$= 0.1184$$

$$y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.0665 + \frac{1}{6}(0.0833 + 2\times 0.1004 + 2\times 0.1008 + 0.1184)$$

$$\boxed{y(0.2) = 1.1672}$$

Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, estimate the value of $y(0.2)$ with $h = 0.1$ using Runge Kutta method of order 4

Stage 1

$f(x, y) = x + y^2$

$x_0 = 0 \quad y_0 = 1 \quad x_1 = x_0 + h$

$= 0 + 0.1$

$x_1 = 0.1$

$f(x_0\, y_0) = 0 + 1^2$

$f(x_0\, y_0) = 1$

$k_1 = h f(x_0\, y_0) = 0.1 \times 1 = 0.1$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 f(0.05, 1.05)$

$= 0.1 \left(0.05 + (1.05)^2\right)$

$= 0.1152$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1 f(0.05, 1.0576)$

$= 0.1 \left(0.05 + (1.0576)^2\right)$

$= 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0.1, 1.1168)$

$= 0.1 (0.1 + (1.1168)^2)$

$= 0.1347$

$y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

stage 2 $= 1.1164$

$x_0 = 0.1 \quad y_0 = 1.1164 \quad x_1 = x_0 + h$

$x_1 = 0.1 + 0.1$

$x_1 = 0.2$

$f(x_0, y_0) = 0.1 + (1.1164)^2$

$= 1.3463$

$k_1 = hf(x_0, y_0) = 0.1 \times 1.3463 = 0.1346$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.1 f(0.15, 1.1837)$

$= 0.1 (0.15 + (1.1837)^2)$

$= 0.1551$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.1 f(0.15, 1.1939)$

$= 0.1 (0.15 + (1.1939)^2)$

$= 0.1575$ //

$k_4 = hf(x_0+h, y_0+k_3)$

$= 0.1 f(0.2, 1.2739)$

$= 0.1 (0.2 + (1.2739)^2)$

$= 0.1823$

$y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.1164 + \frac{1}{6}(0.1346 + 2 \times 0.1551 + 2 \times 0.1575 + 0.1823)$

$= \underline{1.2734}$

solve $(y^2-x^2)dx = (y^2+x^2)dy$ for $x = 0(0.2)(0.4)$ given that $y=1$, at $x=0$ initially by applying runge kutta method of order 4

$\frac{dy}{dx} = \frac{y^2-x^2}{(y^2+x^2)}$

$\frac{y^2-x^2}{(y^2+x^2)}$

$(a+b)(a-b)$

$a^2 - ab + ab - b^2$

$(y+x)(y+x)$

$y^2 + yx + xy + x^2$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.2$

$f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

$f(x,y) = 1$

$x_1 = x_0 + h$

$= 0 + 0.2$

$= 0.2$

$1 + \frac{0.2}{2}$

To find $y(x_1) = y(0.2)$

$k_1 = h f(x_0, y_0) = 0.2 \times 1 = 0.2$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= h f(0.1, 1.1)$

$= 0.2 \times \left(\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right)$

$= 0.1967$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2 f(0.1, 1.0984)$

$= 0.2 \left(\frac{(1.0984)^2 - (0.1)^2}{(1.0984)^2 + (0.1)^2}\right)$

$= 0.1967$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.2 f(0.2, 1.1967)$

$= 0.2 \left(\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right)$

$= 0.1891$

$y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.2 + 2 \times 0.1967 + 2 \times 0.1962 + 0.1891)$

$y(0.2) = 1.19598$

Stage 2

$x_0 = 0.2 \quad y_0 = 1.19598$

$x_1 = x_0 + h$

$x_1 = 0.2 + 0.2$

$x_1 = 0.4$

$y(0.4) = 1.37523$

$f(x,y) = \frac{y^2 - x^2}{(y^2 + x^2)}$

$= \frac{(1.19598)^2 - (0.2)^2}{(1.19598)^2 + (0.2)^2}$

$= 0.9456$