

Module 3: Vector space and Linear Transformations:

Introduction:—

Group:- A non empty set equipped with one binary operation $*$ is called group if $*$ satisfies following Postulates (or axioms).

1. Closure Property: $\forall a \text{ and } b \in G \Rightarrow a * b \in G$.
2. Associative Property: $\forall a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$
3. Existence of identity:

There exists an element $e \in G$ such that
$$a * e = a = e * a$$

The element e is called the identity.

4. Existence of inverse:

Each element of G possesses inverse, i.e. $\forall a \in G \Rightarrow$ there exists an element $b \in G$ such that

$$a * b = e = b * a$$

Then b is called inverse of a and we write $b = a^{-1}$

$$\therefore a * a^{-1} = e = a^{-1} * a$$

* A group with commutative property is known as abelian group or commutative group.

i.e. $\forall a, b \in G$, if $a * b = b * a$.

$\therefore$ ' $*$ ' is Commutative on G .

Eg:- i) $(I, +)$ is an abelian group as $(I, +)$ is a group and also $\forall a, b \in I$.

ii) $(\mathbb{Q}_0, \cdot)$, $(\mathbb{R}_0, \cdot)$, $(\mathbb{C}_0, \cdot)$ are examples of abelian group.

Rings:-

Suppose R is a non-empty set equipped with two binary operation called addition and multiplication and denoted by '+' and '•' respectively.

Then the algebraic structure $(R, +, \cdot)$ is known as ring if the following axioms are satisfied.

1) $(R, +)$ is an abelian group.

i) Closure property for '+'

$$\forall a, b \in R \Rightarrow a+b \in R.$$

ii) Associative property for '+'

$$\forall a, b, c \in R \Rightarrow a+(b+c) = (a+b)+c$$

iii) Existence of identity element for '+'

$$\forall a \in R \exists 0 \in R \ni a+0 = a = 0+a$$

0 is known as additive element or Zero element of the ring.

iv) Existence of inverse element for '+'

$$\forall a \in R \exists -a \in R.$$

$$\text{such that } a+(-a) = 0 = -a+a.$$

$-a$ is known as additive inverse of a .

2) $(R, \cdot)$ is a semi group.

i) Closure property for '•'

$$\forall a, b \in R \Rightarrow a \cdot b \in R.$$

ii) Associative property for '•'

$$\forall a, b, c \in R \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3) Multiplication '•' distributes over addition '+' from left and also from right.

$$\forall a, b, c \in R \quad \text{i) } a \cdot (b+c) = a \cdot b + a \cdot c$$

$$\text{ii) } (a+b) \cdot c = a \cdot c + b \cdot c$$

Eg:- $(\mathbb{I}, +, \cdot)$ is a ring.

* Commutative Ring:-

Ring $(R, +, \cdot)$ is a commutative ring, if $\forall a, b \in R, a \cdot b = b \cdot a$.

Eg:- $(\mathbb{I}, +, \cdot)$.

* Ring With Unity:-

Ring $(R, +, \cdot)$ is known as ring with unity if $\forall a \in R \exists 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$.

* Commutative Ring with Unity:-

Ring $(R, +, \cdot)$ is commutative ring with unity, if $\forall a, b \in R \Rightarrow a \cdot b = b \cdot a$ and $\forall a \in R \exists 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$.

Field:-

A commutative ring with unity in which every non-zero element possesses their multiplicative inverse is known as field.

i.e. $(F, +, \cdot)$ is a field if

- (i) $(F, +)$ is an abelian group.
- (ii) $(F, \cdot)$ is an abelian group.
- (iii) Multiplication distributes over addition from the left and also from the right.

Eg:- $(\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$ are examples of fields.

* Note :-

If $(F, +, \cdot)$ is a field then F contains at least two elements, zero element and unit element i.e additive identity element and multiplicative identity element.

Vector Spaces:-

Let F be a field. A vector space over F , is a non-empty set V together with two operations (called addition and scalar multiplication) such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$, and it satisfies the following conditions:

- I. (i) $u+(v+w) = (u+v)+w$, for all $u, v, w \in V$
(Associativity)
- (ii) $u+v = v+u$, for all $u, v \in V$ (Commutativity)
- (iii) There exists an element $0 \in V$ such that
 $u+0 = u = 0+u$, for all $u \in V$ (Existence of identity element)
- (iv) For each $u \in V$, there exists a unique element
 $-u \in V$ such that $u+(-u) = 0 = (-u)+u$
(Existence of additive inverse)

II. There is an external composition in V over F called scalar multiplication.

$$\text{i.e. } \forall \alpha \in F \text{ and } u \in V \Rightarrow \alpha \cdot u \in V$$

In other words V is closed with respect to scalar multiplication.

III. The two compositions, i.e. vector addition and scalar multiplication satisfy the following postulates,

$\forall a, b \in F$ and $v, w \in V$

(i) $(a+b) \cdot v = a \cdot v + b \cdot v$

(ii) $a \cdot (b \cdot v) = (ab) \cdot v$

(iii) $a \cdot (v+w) = a \cdot v + a \cdot w$

(iv) $1 \cdot (v) = v$.

* Vectors in R^n :-

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 & the set of all n -tuples of real numbers, denoted by R^n , is called Euclidean n -space.

Eg:- The ordered pair $(8, -3)$ belongs to R^2 ; it is a 2-tuple of dimension two.

The ordered triple $(7, 3, 6)$ belongs to R^3 ; it is a 3-tuple of dimension three.

Examples or Problems :-

1) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Solⁿ:- Let V denote the set of all coplanar vectors and R be the field of real numbers.

$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x, y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group.

i) Associativity:-

We know that for all $u, v, w \in V$

$$(u+v)+w = u+(v+w).$$

ii) Commutativity:-

We know that for all $u, v \in V$

$$u+v = v+u$$

iii) Existence of additive identity:-

For every vector $u \in V$ there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$.

iv) Existence of additive inverse:-

For every vector $u \in V$ there exists a vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$.

Thus V is an abelian group with respect to vector addition.

II) Scalar multiplication in V .

i) For $u, v \in V$ and $\alpha \in \mathbb{R}$ we have

$$\alpha(u+v) = \alpha u + \alpha v$$

ii) For $u \in V$ and $a, b \in \mathbb{R}$, we have

$$(a+b)u = au + bu.$$

iii) For $u \in V$ and $a, b \in \mathbb{R}$, we have

$$a(bu) = (ab)u$$

iv) For $u \in V$ and $1 \in \mathbb{R}$, we have

$$1u = u.$$

Thus set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.
 $\therefore V$ is a vector space.

Q) Prove that the set C of all complex numbers (i.e. the set of all ordered pairs of real numbers) is a vector space over the field R of all real numbers where vector addition is defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in C$$

and scalar multiplication is defined by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2), \text{ for all } \alpha \in R.$$

Solⁿ:- We observe that C is closed under vector addition and under scalar multiplication.

I. $(C, +)$ is an abelian group.

(i) Associativity:

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in C$, we have

$$\begin{aligned} (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) \end{aligned}$$

(ii) Commutativity:

For all $(x_1, x_2), (y_1, y_2) \in C$, we have

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) \end{aligned}$$

(iii) Existence of identity:

For $(x_1, x_2) \in C$, there exists $(0, 0) \in C$ such that

$$\begin{aligned} (x_1, x_2) + (0, 0) &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2) \end{aligned}$$

$$(0 + 0) + (x_1, x_2) = (0 + x_1, 0 + x_2) = (x_1, x_2)$$

$$\therefore (x_1, x_2) + (0, 0) = (x_1, x_2) = (0, 0) + (x_1, x_2),$$

(iv) Existence of inverse :

For any $(x_1, x_2) \in G$, there exists $(-x_1, -x_2) \in G$ such that
 $(x_1, x_2) + (-x_1, -x_2) = (0, 0) = (-x_1, -x_2) + (x_1, x_2)$.

Thus C is an abelian group with respect to vector addition.

II. Properties of Scalar multiplication in C .

$$\begin{aligned} \text{(i)} \quad \alpha [(x_1, x_2) + (y_1, y_2)] &= \alpha (x_1 + y_1, x_2 + y_2) \\ &= (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2) \end{aligned}$$

Thus $\alpha [(x_1, x_2) + (y_1, y_2)] = \alpha(x_1, x_2) + \alpha(y_1, y_2)$
for all $(x_1, x_2), (y_1, y_2) \in G, \alpha \in \mathbb{R}$.

$$\begin{aligned} \text{(ii)} \quad (a+b)(x_1, x_2) &= ((a+b)x_1, (a+b)x_2) \\ &= (ax_1 + bx_1, ax_2 + bx_2) \\ &= (ax_1, ax_2) + (bx_1, bx_2) \\ &= a(x_1, x_2) + b(x_1, x_2) \end{aligned}$$

Thus $(a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$ for all
 $(x_1, x_2), (y_1, y_2) \in G$ & $a, b \in \mathbb{R}$.

$$\begin{aligned} \text{(iii)} \quad a(b(x_1, x_2)) &= a(bx_1, bx_2) = (abx_1, abx_2) \\ a(b(x_1, x_2)) &= (ab)(x_1, x_2) \end{aligned}$$

$$\text{(iv)} \quad 1 \cdot (x_1, x_2) = (x_1, x_2) \text{ for all } (x_1, x_2) \in G$$

Thus the set G satisfies all the properties of vector space.

Hence G is a vector space over $\mathbb{R}$.

3) Show that the set V of all real valued continuous functions of x defined on interval $[0,1]$ is a vector space over the field R of real numbers with respect to vector addition and scalar multiplication defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \text{ for all } f_1, f_2 \in V$$

$$(\alpha f_1)(x) = \alpha f_1(x), \text{ for all } \alpha \in R, f_1 \in V.$$

Solⁿ:-

I. $(V, +)$ is an abelian group.

(i) Associativity:

Let $f_1, f_2, f_3 \in V$ be arbitrary.

$$\begin{aligned} [(f_1 + f_2) + f_3](x) &= (f_1 + f_2)(x) + f_3(x) \\ &= [f_1(x) + f_2(x)] + f_3(x) \\ &= f_1(x) + [f_2(x) + f_3(x)] \\ &= f_1(x) + (f_2 + f_3)(x) \\ &= [f_1 + (f_2 + f_3)](x) \end{aligned}$$

$$(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3).$$

(ii) Commutativity:

Let $f_1, f_2 \in V$ be arbitrary.

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ &= f_2(x) + f_1(x) \\ &= (f_2 + f_1)(x), \text{ for all } x. \end{aligned}$$

$$f_1 + f_2 = f_2 + f_1.$$

(iii) Existence of Identity:

Define a function 0 such that $0(x) = 0$ (real number), for all $x \in [0,1]$.

Also, 0 is a continuous function and belongs to V

$$(0 + f_1)(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$

$$(f_1 + 0)(x) = f_1(x) + 0(x) = f_1(x)$$

$$0 + f_1 = f_1 + 0 = f_1$$

Thus, the function 0 defined above is an additive identity element of V .

(iv) Existence of Inverse:

For a function f_1 , the function $-f_1$ defined by $(-f_1)(x) = -f_1(x)$, is called additive

inverse, as $[f_1 + (-f_1)](x) = f_1(x) + (-f_1)(x)$
 $= f_1(x) - f_1(x) = 0 = 0(x)$

$$\text{Similarly, } [-f_1 + f_1](x) = 0(x).$$

Thus the set V is an abelian group under addition.

II. Properties of Scalar Multiplication in V .

(i) For $f_1, f_2 \in V$ and $\alpha \in R$, we have

$$\begin{aligned} [\alpha(f_1 + f_2)]x &= \alpha[(f_1 + f_2)(x)] \\ &= \alpha[f_1(x) + f_2(x)] \\ &= \alpha f_1(x) + \alpha f_2(x) \\ &= (\alpha f_1)x + (\alpha f_2)x \\ &= (\alpha f_1 + \alpha f_2)x \end{aligned}$$

$$\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$$

(ii) For $f_1 \in V$ and $a, b \in R$.

$$\begin{aligned} [(a+b)f_1](x) &= (a+b)f_1(x) \\ &= a f_1(x) + b f_1(x) \\ &= (a f_1)x + (b f_1)x \\ &= (a f_1 + b f_1)(x) \end{aligned}$$

$$(a+b)f_1 = a f_1 + b f_1$$

(iii) For $f_1 \in V$ and $a, b \in R$,

$$\begin{aligned} [a(b f_1)]x &= a[(b f_1)x] \\ &= a[b f_1(x)] \\ &= (ab)f_1(x) \\ &= [(ab)f_1]x \end{aligned}$$

$$a(b f_1) = (ab) f_1$$

(iv) For $f_1 \in V$ and $1 \in R$, We have

$$(1, f_1)x = 1 \cdot f_1(x) = f_1(x)$$

$$1 f_1 = f_1$$

$\therefore$ Thus V satisfies all the properties of a vector space and hence V is a vector space.

4) Show that the set of all matrices of the type $m \times n$ where m and n are fixed positive integers is a vector space over R with respect to matrix addition and multiplication of matrix by a scalar (i.e., a real number)

Solⁿ: Let M denotes the set of all matrices of type $m \times n$.

I. $(M, +)$ is an abelian group.

(i) Associativity:

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{m \times n}$ be three matrices belonging to set M .

$$(A+B)+C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$$

$$= [a_{ij} + b_{ij}] + [c_{ij}]$$

$$= [(a_{ij} + b_{ij}) + c_{ij}]$$

$$= [a_{ij} + (b_{ij} + c_{ij})]$$

$$= [a_{ij}] + [b_{ij} + c_{ij}]$$

$$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$$

$$= A + (B+C)$$

(ii) Commutativity:

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices belonging to set M .

$$\begin{aligned}
 A+B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\
 &= [b_{ij} + a_{ij}] \\
 &= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} \\
 &= B + A.
 \end{aligned}$$

(iii) Existence of Identity:

Let $A = [a_{ij}]_{m \times n} \in M$ be arbitrary.

We know that a zero matrix (null matrix) of type $m \times n$ also belongs to the set M and is denoted by O .

$$\text{Now, } A + O = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$$

$$\text{Similarly, } O + A = A.$$

(iv) Existence of Inverse:

If $A = [a_{ij}]_{m \times n}$ belongs to the set M , then

$-A = [-a_{ij}]_{m \times n}$ also belongs to the set M .

$$\begin{aligned}
 \text{Now, } A + (-A) &= [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] \\
 &= [0] = O
 \end{aligned}$$

$$\text{Similarly, } (-A) + A = O$$

Thus the set M is an abelian group under addition.

II. Properties of Scalar multiplication in M .

(i) Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n} \in M$ and $\alpha \in R$, then

$$\alpha(A+B) = \alpha([a_{ij}] + [b_{ij}])$$

$$= \alpha[a_{ij} + b_{ij}]$$

$$= [\alpha(a_{ij} + b_{ij})]$$

$$= [\alpha \cdot a_{ij} + \alpha \cdot b_{ij}] \quad [\because \text{Multiplication is distributive in } R]$$

$$= [\alpha \cdot a_{ij}] + [\alpha \cdot b_{ij}]$$

$$= \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B.$$

(ii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$, then

$$\begin{aligned}(a+b)A &= (a+b)[a_{ij}] \\ &= [(a+b)a_{ij}] \\ &= [a \cdot a_{ij} + b \cdot a_{ij}] \\ &= a[a_{ij}] + b[a_{ij}] \\ &= aA + bA.\end{aligned}$$

(iii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$ then

$$\begin{aligned}a(bA) &= a(b[a_{ij}]) \\ &= a([ba_{ij}]) \\ &= [(ab)a_{ij}] \\ &= (ab)[a_{ij}] \\ &= (ab)A \\ a(bA) &= (ab)A.\end{aligned}$$

(iv) Let $A = [a_{ij}]_{m \times n} \in M$, then

$$1 \cdot A = 1 \cdot [a_{ij}] = [1 \cdot a_{ij}] = [a_{ij}] = A.$$

Thus M satisfies all the properties of vector space and hence M is a vector space over R .

5) If V is an abelian group of positive real numbers for multiplication. Define scalar multiplication in V by $a \cdot x = x^a$ where $a \in R$, $x \in V$. Show that V is a vector space over R .

Solⁿ: - Since $V = R^+$ is an abelian group for multiplication.

There are two operations

(i) $\times$, i.e multiplication in R^+

(ii) $\cdot$, i.e Scalar multiplication in R^+

$$a \in \mathbb{R}, x \in \mathbb{R}^+$$

$$a \cdot x = x^a$$

and $x^a \in \mathbb{R}^+$ as $x \in \mathbb{R}^+$, x^a is always positive.

Eg:- Let $a = -3$
 $-3 \cdot x = x^{-3} = \frac{1}{x^3}$ is a positive real number.

Hence x^{-3} is an element of $\mathbb{R}^+$.

Now,

I. $(V, \times)$ is an abelian group.

II. $\forall a \in \mathbb{R}, x \in V \Rightarrow a \cdot x = x^a \in V$.

III. (i) $(a+b) \cdot x = x^{a+b} = x^a \times x^b = (a \cdot x) \times (b \cdot x)$

$$\begin{aligned} \text{(ii) } a \cdot (x_1 \times x_2) &= (x_1 \times x_2)^a \\ &= x_1^a \times x_2^a \\ &= (a \cdot x_1) \times (a \cdot x_2) \end{aligned}$$

$$\begin{aligned} \text{ciii) } a \cdot (b \cdot x) &= (b \cdot x)^a \\ &= (x^b)^a = x^{ba} \\ &= x^{ab} \quad (\because \mathbb{R} \text{ is commutative for multiplication}) \\ &= ab \cdot (x) \end{aligned}$$

$$\text{(iv) } 1 \cdot (x) = x^1 = x$$

Hence $(V, \times, \bullet)$ is a vector space.

Subspaces :-

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V .

Eg:- Let $W_1 = \{(a, 0, 0) : a \in \mathbb{R}\}$
 $W_2 = \{(a, b, 0) : a, b \in \mathbb{R}\}$.

Here W_1 is a subspace of W_2 . Also W_1 and W_2 are subspace of $\mathbb{R}^3$.

* Necessary and Sufficient conditions for a subspace:
(Only statement, no proof).

Theorem 1 : W is a subspace of $V(F)$ iff

- i) W is non empty.
- ii) W is closed under vector addition.
i.e. $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.
- iii) W is closed under scalar multiplication.
i.e. $\forall a \in F$ and $w \in W \Rightarrow a.w \in W$.

Theorem 2 : W is a subspace of $V(F)$ iff

- i) W is non empty.
- ii) $\forall a, b \in F$ and $v, w \in W \Rightarrow$
 $a.v + b.w \in W$.

Problems:-

1) Show that W is a subspace of $V(R)$ where $W = \{f: f(9) = 0\}$

Solⁿ:- Since $0 \in W$ as $0(9) = 0$

So W is non empty set.

Let $f, g \in W$.

i.e. $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in R$

$$(a.f + b.g)(9) = a.f(9) + b.g(9) = a.0 + b.0 = 0$$

Hence $a.f + b.g \in W$.

By theorem (2), W is a subspace of $V(R)$.

2) Show that W is a subspace of $V(R)$ where $W = \{f: f(2) = f(1)\}$

Solⁿ:- $0 \in W$ since $0(2) = 0 = 0(1)$

hence W is a non empty set.

Let $f, g \in W$ then $f(2) = f(1)$ & $g(2) = g(1)$

then $\forall a, b \in R$.

$$\begin{aligned}(a.f + b.g)(2) &= a.f(2) + b.g(2) = a.f(1) + b.g(1) \\ &= (a.f + b.g)(1)\end{aligned}$$

hence $a.f + b.g \in W$.

So by theorem 2, W is a subspace of $V(R)$.

3) Let $V = R^3$ be the Euclidean 3-space. Let

$W = \{(x, y, z) : ax + by + cz = 0; x, y, z \in R\}$, a, b, c being real numbers. Show that W is a subspace of V .
(OR)

Show that any plane passing through the origin is a subspace of $\mathbb{R}^3$.

Solⁿ: - Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two elements of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$.

Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

For $\alpha \in \mathbb{R}$, $\alpha u + v = \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$\begin{aligned} \alpha u + v &= (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2) \\ &= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \rightarrow (1) \end{aligned}$$

Where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2 \in \mathbb{R}$.

Now

$$\begin{aligned} a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha z_1 + z_2) &= \\ \alpha(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) &= \\ = \alpha(0) + 0 = 0 &\rightarrow (2) \end{aligned}$$

From (1) and (2), we have

$$\alpha u + v \in W.$$

Thus, W is a subspace of V .

4) Let V be the vector space of all square matrices over $\mathbb{R}$.

Determine which of the following are subspaces of V .

$$(i) W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \quad (ii) W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$(iii) W = \{A : A \in V \text{ and } A \text{ is singular}\} \quad (iv) W = \{A : A \in V, A^2 = A\}$$

Soln:- (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ x_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ x_2 & 0 \end{bmatrix}$ be any two elements of W .

If $a, b \in \mathbb{R}$ then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} x_1 & y_1 \\ x_1 & 0 \end{bmatrix} + b \begin{bmatrix} x_2 & y_2 \\ x_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 & ay_1 \\ ax_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bx_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ ax_1 + bx_2 & 0 \end{bmatrix} \end{aligned}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ x & 0 \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2, ax_1 + bx_2 \in \mathbb{R}.$$

$$\therefore aA + bB \in W.$$

Thus, W is a sub-space of V .

ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements of W .

If $a, b \in \mathbb{R}$, then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ay_1 + by_2 \end{bmatrix} \end{aligned}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2 \in \mathbb{R}.$$

$$\therefore aA + bB \in W.$$

Thus W is a subspace of V .

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A| = 0$ and $|B| = 0$, therefore $A, B \in W$.

If $a, b \in \mathbb{R}$ are non zero, then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \end{aligned}$$

Also, $|aA_1 + bA_2| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$, as none of a and b is zero.

$\therefore aA_1 + bA_2 \notin W$.

Thus, W is not a sub-space of V .

(iv) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$

$\therefore A \in W$.

$$\text{Now } A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(A+A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A + A$$

$\therefore (A+A)$ is not a member of the set W .

$\Rightarrow W$ is not closed under addition.

Hence W is not a sub-space of V .

5) Let V be the vector space of all real valued continuous functions over $\mathbb{R}$. Show that the set W of solutions of differential equations $5 \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0$ is a subspace of V .

Solⁿ: - Let $y_1, y_2 \in W$. Then y_1, y_2 are solutions of differential equations.

$$5 \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0 \rightarrow (1)$$

$$\text{i.e. } 5 \frac{d^2y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 = 0 ; 5 \frac{d^2y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 = 0.$$

Let $a, b \in \mathbb{R}$ be arbitrary.

$$5 \frac{d^2}{dx^2} (ay_1 + by_2) - 7 \frac{d}{dx} (ay_1 + by_2) + 2(ay_1 + by_2)$$

$$= 5 \frac{d^2}{dx^2} (ay_1) - 7 \frac{d}{dx} (ay_1) + 2(ay_1) +$$

$$5 \frac{d^2}{dx^2} (by_2) - 7 \frac{d}{dx} (by_2) + 2(by_2).$$

$$= a \left[5 \frac{d^2y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 \right] + b \left[5 \frac{d^2y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 \right]$$

$$= a \cdot 0 + b \cdot 0 = 0.$$

Thus $ay_1 + by_2$ is a solution of (1) & so $ay_1 + by_2 \in W$.

$\therefore W$ is a subspace of V .

Linear Combination :-

Let V be a vector space over a field F and let

$$v_1, v_2, \dots, v_n \in V.$$

Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ in V , where $a_i \in F$ is called a linear combination of $v_1, v_2, \dots, v_n$.

Linear Dependence :-

Let V be a vector space over the field F . The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent over F , if $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ not all zero but linear combination is zero.

$$\text{i.e. } a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0.$$

but all $a_i \neq 0$, where $i \in N$.

Linear Independence :-

Let V be a vector space over the field F . The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent over F , if $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ such that

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \Rightarrow \text{all } a_i = 0 \text{ where } i \in N.$$

Linear Span :-

Let S be a subset of the vector space V over the field F . The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$.
If $S = \phi$, then $L(S) = 0$.

Basis or Base of Vector space V :-

Let V be a vector space over the field F . The set of vectors $\{v_1, v_2, \dots, v_n\}$ is called a basis of V , if

- (i) $v_1, v_2, \dots, v_n$ are linearly independent,
- (ii) $v_1, v_2, \dots, v_n$ span V . i.e. each vector of V can be uniquely expressed as linear combination of $v_1, v_2, \dots, v_n$.

Dimension of a vector space V :-

Number of elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V$.

If V contains a basis with n elements then the $\dim V = n$.

Note :-

- 1) The vector space $\{0\}$ is defined to have $\dim 0$, since empty set ϕ is independent and generates $\{0\}$.

$$\begin{aligned}\therefore \dim \{0\} &= \text{Number of elements in } \phi \\ &= 0 \text{ [Since no element is in } \phi \text{]}\end{aligned}$$

- 2) When a vector space is not of finite dimension, it is said to be of infinite dimension.

Problems :-

1) Express the vector $v = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, -1, 1)$ in the vector space $R^3(R)$.

Soln:- Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$; $a_1, a_2, a_3 \in R$

$$(1, -2, 5) = a_1(1, 1, 1) + a_2(1, 2, 3) + a_3(2, -1, 1)$$

$$(1, -2, 5) = (a_1 + a_2 + 2a_3, a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$$

Equating the corresponding elements, we get

$$a_1 + a_2 + 2a_3 = 1; a_1 + 2a_2 - a_3 = -2; a_1 + 3a_2 + a_3 = 5$$

Solving above equations, we get

$$a_1 = -6, a_2 = 3, a_3 = 2$$

$$\text{Hence } (1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) + 2(2, -1, 1).$$

2) Write the vector $v = (1, 3, 9)$ as a linear combination of the vectors $u_1 = (2, 1, 3)$, $u_2 = (1, -1, 1)$, $u_3 = (3, 1, 5)$

(Another method of solving)

Soln:- Let $v = x u_1 + y u_2 + z u_3$

$$(1, 3, 9) = x(2, 1, 3) + y(1, -1, 1) + z(3, 1, 5)$$

$$\Rightarrow 2x + y + 3z = 1; x - y + z = 3; 3x + y + 5z = 9$$

Consider $AX = B$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix,

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 2 & 1 & 3 & : & 1 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 3 & 1 & : & -5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 0 & -2 & : & -20 \end{bmatrix}$$

$\therefore \text{Rank } [A] = \text{Rank } [A:B] = 3 = \text{Number of unknowns}$

$\therefore$ The system of linear equations is consistent and possesses a unique soln.

$$x + y + z = 3; \quad y + z = 5; \quad -2z = -20$$

$$\therefore z = 10, \quad y = -5, \quad x = -12$$

$$\therefore (1, 3, 9) = -12(2, 1, 3) - 5(1, -1, 1) + 10(3, 1, 5).$$

3) Write the vector $v = (4, 2, 1)$ as a linear combination of the vectors $u_1 = (1, -3, 1)$, $u_2 = (0, 1, 2)$, $u_3 = (5, 1, 3)$.

Solⁿ:- Let $v = x u_1 + y u_2 + z u_3$

$$(4, 2, 1) = x(1, -3, 1) + y(0, 1, 2) + z(5, 1, 3)$$

$$\Rightarrow x + 0y + 5z = 4$$

$$-3x + y + z = 2$$

$$x + 2y + 3z = 1$$

$$\text{Let } AX = B. \text{ Where } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Consider, } [A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & -2 & -3 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & -34 & -31 \end{array} \right]$$

$$\text{Rank}[A] = 2 \neq \text{Rank}[A:B] = 3.$$

Since $\text{Rank}[A] \neq \text{Rank}[A:B]$. Hence the system of linear equations is inconsistent.

i.e. solⁿ does not exist.

$\therefore v$ cannot be expressed as linear combination of the vectors u_1, u_2, u_3 .

4) For what value of k (if any) the vector $v = (1, -2, k)$ can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$ in $R^3(R)$.

Solⁿ:- Since vector $v = (1, -2, k)$ is a linear combination of $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$; therefore there exist scalars a and b such that

$$v = av_1 + bv_2$$

$$(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$$

$$(1, -2, k) = (3a + 2b, -b, -2a - 5b)$$

Equating the corresponding elements,

$$3a + 2b = 1, \quad -b = -2, \quad -2a - 5b = k$$

$$3a + 4 = 1, \quad b = 2, \quad -2(-1) - 5(2) = k$$

$$a = -1$$

$$2 - 10 = k$$

$$\therefore k = -8.$$

5) Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$ in R^3 so that $w \in \text{Span}(u, v)$.

Solⁿ:- Let $w = xu + yv$

$$(a, b, c) = x(1, -3, 2) + y(2, -1, 1)$$

$$x + 2y = a; \quad -3x - y = b; \quad 2x + y = c$$

Let $AX = B$.

$$\text{Consider } [A:B] = \left[\begin{array}{cc|c} 1 & 2 & a \\ -3 & -1 & b \\ 2 & 1 & c \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 5 & : & b+3a \\ 0 & -3 & : & c-2a \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{5}$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & -3 & : & c-2a \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & 0 & : & \frac{5c-a+3b}{5} \end{bmatrix}$$

$$\text{Rank } [A] = 2$$

The system of linear equations will be consistent if
 $\text{Rank } [A:B] = 2$

$$\text{So, } \frac{5c-a+3b}{5} = 0 \Rightarrow a-3b-5c=0$$

i.e. w is linear combination of u and v iff

$$a-3b-5c=0.$$

6) Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Soln:- Let $A = a_1 B + a_2 C + a_3 D$; $a_1, a_2, a_3 \in \mathbb{R} \rightarrow (1)$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

Equating the corresponding elements,

$$a_1 + a_2 + a_3 = 3 ; a_1 + a_2 - a_3 = -1 ; -a_2 = 1 ; -a_1 = -2$$

Solving the above equations,

$$a_1 = 2, a_2 = -1, a_3 = 2$$

$\therefore$ (1) $\Rightarrow$

$$A = 2B - C - 2D$$

(or)
$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the required linear combination of A.

7) Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Soln:- Let $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$;
 $a_1, a_2, a_3 \in \mathbb{R}$.

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 2a_3 & -3a_1 + 3a_3 \\ 2a_1 + 2a_2 & a_2 + 5a_3 \end{bmatrix}$$

Equating the corresponding elements $2 = 2a_3 \rightarrow$ (1)

$$0 = -3a_1 + 3a_3 \rightarrow (2)$$

$$4 = 2a_1 + 2a_2 \rightarrow (3)$$

$$-5 = a_2 + 5a_3 \rightarrow (4)$$

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$0 = -3a_1 + 3a_3 \Rightarrow a_1 = 1$$

$$4 = 2a_1 + 2a_2 \Rightarrow a_2 = 1$$

But $a_1 = 1, a_2 = 1, a_3 = 1$ do not satisfy eqⁿ (4)

Thus equations (1), (2) & (3) have no solution.

Hence, the given matrix cannot be expressed as a linear combination of A, B, C.

Note:-

* Two vectors v_1 and v_2 are linearly dependent iff one of them is a multiple of the other.

Ex:- Determine whether or not the vectors v_1, v_2 are linearly dependent.

i) $v_1 = (1, 3, 9)$ $v_2 = (2, 4, 1)$.

→ No vector is the multiple of the other hence v_1 & v_2 are not linearly dependent.

ii) $v_1 = (3, 4)$ $v_2 = (6, 8)$.

→ $v_2 = 2v_1$, hence v_1 and v_2 are linearly dependent vectors.

8) Determine whether the vectors $v_1 = (1, 2, 3)$ $v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Soln:- $xv_1 + yv_2 + zv_3 = 0$

$$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$$

$$x + 3y + 2z = 0; \quad 2x + y + 5z = 0; \quad 3x + 7y + 8z = 0.$$

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 3R_1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{aligned} R_2 &\rightarrow -R_2 \\ R_3 &\rightarrow -\frac{R_3}{2} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5R_2$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank $[A] = 3 = \text{Number of unknowns}$.

$\Rightarrow$ Homogeneous system of linear equations possesses unique solution.

i.e. $x=0, y=0, z=0$.

So vectors v_1, v_2, v_3 are linearly independent.

9) Determine whether the vectors $v_1 = (1, 9, 3), v_2 = (2, 5, 4), v_3 = (0, 0, 0)$ are linearly dependent or linearly independent.

Soln:- vector v_1, v_2, v_3 are linearly dependent as linearly independent set of vectors can not contain zero vector.

10) Determine whether the vectors $v_1 = (1, 4, 9), v_2 = (3, 1, 4), v_3 = (9, 3, 12)$ are linearly dependent or linearly independent.

Soln:- $xv_1 + yv_2 + zv_3 = 0$.

$$x(1, 4, 9) + y(3, 1, 4) + z(9, 3, 12) = (0, 0, 0)$$

$$x + 3y + 9z = 0; \quad 4x + y + 3z = 0; \quad 9x + 4y + 12z = 0$$

Consider $AX = 0$. Where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$R_3 \rightarrow R_3 - 9R_1$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{R_2}{11}$$

$$R_3 \rightarrow -\frac{R_3}{23}$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $[A] = 2 < 3$ (Number of unknowns)

So one unknown is constant and hence not always zero.

So the vectors are linearly dependent.

[OR]

$$v_3 = 3v_2$$

Vectors v_1, v_2, v_3 are linearly dependent.

11) If u, v, w are linearly independent vectors in $V(F)$, where F is the field of complex numbers, then $\{u+v, v+w, w+u\}$ is a linearly independent set of vectors.

So:- Let $a(u+v) + b(v+w) + c(w+u) = 0$, where $a, b, c \in F$
 $(a+c)u + (a+b)v + (b+c)w = 0 \rightarrow (1)$

Since u, v, w are linearly independent.

$$\therefore \textcircled{1} \Rightarrow \begin{aligned} a+c &= 0 \\ a+b &= 0 \\ b+c &= 0. \end{aligned}$$

$$\therefore a=0, b=0, c=0.$$

$\therefore \{u+v, v+w, w+u\}$ is a linearly independent set of vectors.

12) Let V be the vector space of all 2×3 matrices over $\mathbb{R}$. S.T the matrices $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ form a linearly independent set.

Solⁿ: - Let $aA + bB + cC = 0$ where $a, b, c \in \mathbb{R}$.

$$\Rightarrow a \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a+0b-2c & 4a+5b+3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By equating the corresponding elements

$$\left. \begin{array}{l} 2a+b+4c=0 \\ a+b-c=0 \\ -a-3b+2c=0 \end{array} \right\} \rightarrow \textcircled{1} \quad \left. \begin{array}{l} 3a-2b+c=0 \\ -2a-2c=0 \\ 4a+5b+3c=0 \end{array} \right\} \rightarrow \textcircled{2}$$

On solving the system $\textcircled{1}$ of equations, we observe that the only solution is $a=0, b=0, c=0$. Clearly this solution also satisfies the system $\textcircled{2}$ of equations.

Hence the given set of matrices is linearly independent.

13) Show that the vectors $(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0)$ and $(2, 1, 1, 6)$ are linearly dependent.

14) Determine whether or not each of the following forms a basis, $x_1 = (1, 2, 3), x_2 = (3, -5, 6)$ in $\mathbb{R}^3$.

Solⁿ: - Basis in $\mathbb{R}^3$ must contain exactly 3 elements therefore x_1, x_2 does not form a basis in $\mathbb{R}^3$.

15) Determine whether or not each of the following form a basis $x_1 = (1, 2, 9)$, $x_2 = (2, -3, 4)$, $x_3 = (1, 3, 7)$, $x_4 = (2, 4, 8)$ in $\mathbb{R}^3$.

Soln:- Basis in $\mathbb{R}^3$ must contain exactly 3 elements.
 $\therefore x_1, x_2, x_3, x_4$ does not form a basis in $\mathbb{R}^3$.

16) Determine whether or not each of the following forms a basis, $x_1 = (2, 2, 1)$, $x_2 = (1, 3, 7)$, $x_3 = (1, 2, 2)$ in $\mathbb{R}^3$.

Soln:- Three vectors in $\mathbb{R}^3$ form a basis iff they are linearly independent.

$$x(2, 2, 1) + y(1, 3, 7) + z(1, 2, 2) = (0, 0, 0)$$

$$2x + y + z = 0; \quad 2x + 3y + 2z = 0; \quad x + 7y + 2z = 0.$$

Let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow -R_2 \\ R_3 \rightarrow -R_3 \end{array}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{2}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 11 & 2 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 11R_2}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}$$

Rank $[A] = 3 = \text{Number of unknowns}$. So unique solution and solution is

$$x=0, y=0, z=0.$$

Hence vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

17) Let W be the subspace of $\mathbb{R}^5$, spanned by

$$x_1 = (1, 2, -1, 3, 4), x_2 = (2, 4, -2, 6, 8), x_3 = (1, 3, 2, 2, 6)$$

$$x_4 = (1, 4, 5, 1, 8), x_5 = (2, 7, 3, 3, 9)$$

Find a subset of vectors which forms a basis of W .

Solⁿ: - $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$R_5 \rightarrow R_5 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_3, R_5 \rightarrow R_5 - 3R_3$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

Number of non zero rows = 3

$\therefore \dim W = 3$ and $\{x_1, x_3, x_5\}$ forms a basis in W .

18) V is a vector space of polynomials over $\mathbb{R}$. Find a basis and dimension of the subspace W of V , spanned by the polynomials.

$$x_1 = t^3 - 2t^2 + 4t + 1, x_2 = 2t^3 - 3t^2 + 9t - 1$$

$$x_3 = t^3 + 6t - 5, x_4 = 2t^3 - 5t^2 + 7t + 5.$$

Soln:- $A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - R_1 \\ R_4 &\rightarrow R_4 - 2R_1 \end{aligned} \quad \begin{aligned} R_3 &\rightarrow R_3 - 2R_2 \\ R_4 &\rightarrow R_4 + R_2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non Zero rows of Echelon matrix forms a basis.

Number of non zero rows = 2

$\therefore \dim W = 2$ and $\{x_1, x_2\}$ forms a basis in W .

Linear Transformations: -

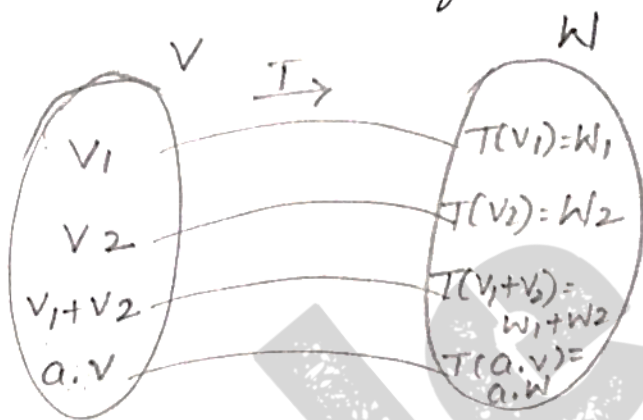
Let V and W be any two subspaces over the field F . A mapping T from V to W is called a linear transformation if,

i) $v_1, v_2 \in V$

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

ii) $\forall a \in F$ and $\forall v \in V$

$$T(a \cdot v) = a \cdot T(v)$$



Problems: -

1) Show that the function $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $T(x, y) = (x+y, x-y, y)$ is a linear transformation.

Soln:- Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two vectors belonging to $\mathbb{R}^2$.

$$\therefore T(u+v) = T(x_1+x_2, y_1+y_2)$$

$$= (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2, y_1+y_2)$$

$$= \{ (x_1+y_1) + (x_2+y_2), (x_1-y_1) + (x_2-y_2), (y_1+y_2) \}$$

$$= (x_1+y_1, x_1-y_1, y_1) + (x_2+y_2, x_2-y_2, y_2)$$

$$= T(u) + T(v)$$

Also for $a \in \mathbb{R}$ and $u \in \mathbb{R}^2$, we have

$$T(au) = T(ax_1, ay_1)$$

$$= (ax_1 + ay_1, ax_1 - ay_1, ay_1)$$

$$= a(x_1 + y_1, x_1 - y_1, y_1)$$

$$= a T(u)$$

$\therefore T$ is a linear transformation.

Q) Which of the following functions are linear transformations?

i) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (y, -x, -z)$

ii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (2x - 3y, 7y + 2z)$.

Soln:- i) Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in \mathbb{R}^3$ be arbitrary.

$$T(u) = T(x_1, y_1, z_1) = (y_1, -x_1, -z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (y_2, -x_2, -z_2)$$

$$\begin{aligned}\text{Now, } T(u+v) &= T(x_1+x_2, y_1+y_2, z_1+z_2) \\ &= (y_1+y_2, -x_1-x_2, -z_1-z_2) \\ &= (y_1, -x_1, -z_1) + (y_2, -x_2, -z_2) \\ &= T(u) + T(v)\end{aligned}$$

Also, for any scalar $a \in \mathbb{R}$,

$$\begin{aligned}T(au) &= T(ax_1, ay_1, az_1) \\ &= (ay_1, -ax_1, -az_1) \\ &= a(y_1, -x_1, -z_1) \\ &= aT(u)\end{aligned}$$

Hence, T is a linear transformation.

ii) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in \mathbb{R}^3$ be arbitrary.

$$T(u) = T(x_1, y_1, z_1) = (2x_1 - 3y_1, 7y_1 + 2z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (2x_2 - 3y_2, 7y_2 + 2z_2)$$

$$\begin{aligned}\therefore T(u+v) &= T(x_1+x_2, y_1+y_2, z_1+z_2) \\ &= (2x_1+2x_2-3y_1-3y_2, 7y_1+7y_2+2z_1+2z_2) \\ &= \{(2x_1-3y_1) + (2x_2-3y_2), (7y_1+2z_1) + (7y_2+2z_2)\} \\ &= (2x_1-3y_1, 7y_1+2z_1) + (2x_2-3y_2, 7y_2+2z_2) \\ &= T(u) + T(v)\end{aligned}$$

Also, for any scalar $a \in \mathbb{R}$

$$\begin{aligned}T(au) &= T(ax_1, ay_1, az_1) \\ &= (2ax_1 - 3ay_1, 7ay_1 + 2az_1)\end{aligned}$$

$$= a(2x_1 - 3y_1, 7y_1 + 2x_1)$$

$$= aT(u).$$

Hence, T is a linear transformation.

3) Let $I: V(R) \rightarrow V_2(R)$ be a mapping $f(x) = (3x, 5x)$. Show that f is a linear transformation.

Soln:- i) $f(x_1 + x_2) = (3(x_1 + x_2), 5(x_1 + x_2))$

$$= (3x_1 + 3x_2, 5x_1 + 5x_2)$$

$$= (3x_1, 5x_1) + (3x_2, 5x_2)$$

$$= f(x_1) + f(x_2)$$

ii) $f(a \cdot x) = (3(a \cdot x), 5(a \cdot x))$

$$= (3ax, 5ax)$$

$$= a(3x, 5x)$$

$$= a \cdot f(x).$$

Hence f is a linear transformation.

4) Show that the translation mapping $f: V_2(R) \rightarrow V_2(R)$ defined by $f(x, y) = (x+6, y+2)$ is not linear.

Soln:- $f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + 6, y_1 + y_2 + 2) \rightarrow (1)$

$$f(x_1, y_1) + f(x_2, y_2) = (x_1 + 6, y_1 + 2) + (x_2 + 6, y_2 + 2)$$

$$= (x_1 + x_2 + 12, y_1 + y_2 + 4) \rightarrow (2)$$

$$\text{Equation (1)} \neq \text{Equation (2)}.$$

Hence f is not a linear transformation.

* Kernel of T and Image of T :

Let T be a linear transformation from V to W

Kernel of T or $\text{Ker}(T) = \{v \in V : T(v) = 0, 0 \in W\}$

Remark: - Kernel of T is also known as null space of T .

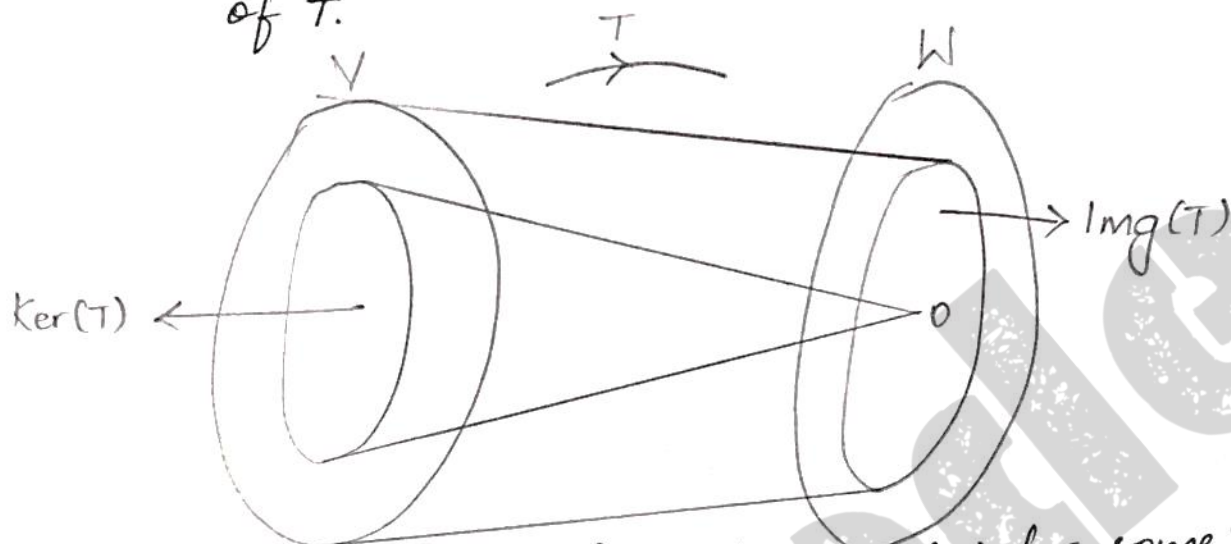


Image of T or $\text{Im}(T) = \{w \in W : w = T(v) \text{ for some } v \in V\}$

Algebra of Linear Transformations: -

* Sum of Linear Transformations: -

Let $T_1: U \rightarrow V$ and $T_2: U \rightarrow V$ be two transformations where U and V are two vector spaces over the field F . We define the sum of T_1 and T_2 by $T_1 + T_2: U \rightarrow V$, such that $(T_1 + T_2)u = T_1(u) + T_2(u), u \in U$.

* Scalar multiplication of Linear Transformations: -

If $a \in F$, the function aT , i.e. product of a linear transformation $T: U \rightarrow V$ with a , and defined by $(aT): U \rightarrow V$, such that $(aT)u = a(T(u))$ for all $u \in U$ is a linear transformation from U into V .

* Composition (Product) of two linear transformations:

Let U, V, W be three vector spaces over a field F . We define $T_2 T_1$, called the composition or product of T_2 and T_1 , by $(T_2 T_1)(u) = T_2(T_1(u))$.

$T_2 T_1$ is also denoted by $(T_2 \circ T_1)$.

Problems:—

1) Let the linear transformations $T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T_2(x, y) = (-2x, y)$. Compute $T_1 T_2$ and $T_2 T_1$.

Soln:— As the range of T_2 is not contained in the domain of T_1 ,

$\therefore T_1 T_2$ is not defined.

Now, $T_2 T_1$ is defined as the range of T_1 is contained in the domain of T_2 .

$$\begin{aligned} T_2 T_1(x, y, z) &= T_2(T_1(x, y, z)) \\ &= T_2(4x, 3y - 2z) \\ &= (-8x, 3y - 2z). \end{aligned}$$

2) Illustrate with the help of an example that there exist linear transformations $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T_2 T_1 = 0$ but $T_1 T_2 \neq 0$.

Soln:— Let us define $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T_1(x, y) = (0, 4x)$

& $T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T_2(x, y) = (x, 0)$

$\therefore T_1 T_2$ and $T_2 T_1$ both are defined.

$$(T_2 T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, 4x) \\ = (0, 0) = 0(x, y)$$

$$T_2 T_1 = 0.$$

$$\text{Also } (T_1 T_2)(x, y) = T_1(T_2(x, y)) = T_1(x, 0) \\ = (0, 4x) \neq 0(x, y)$$

$$T_1 T_2 \neq 0.$$

Matrix of a Linear Transformation:—

Let $T: U \rightarrow V$ be the linear transformation, where U and V are vector spaces over field F .

Let $B = \{u_1, u_2, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively.

Since $T(u_1), T(u_2), \dots, T(u_n) \in V$ and $\{v_1, v_2, \dots, v_m\}$ spans V , each $T(u_i)$ can be expressed as a linear combination of the vectors $v_1, v_2, \dots, v_m$.

$$\text{Let } T(u_1) = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m \\ T(u_2) = a_{12}v_1 + a_{22}v_2 + \dots + a_{m2}v_m$$

$$\dots \dots \dots T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \dots + a_{mn}v_m \text{ where } a_{ij} \in F.$$

The coefficient matrix of this system of equations is

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ - & - & - & - & - \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{bmatrix}$$

The transpose of this matrix is a matrix representation of T , called matrix of T with respect to ordered basis B and B' (or matrix associated with T w.r.to B and B') It is denoted by $[T: B, B']$ and is given by

$$[T: B, B'] = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ \vdots & \vdots & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$$

Problems:-

1) Find the matrix representing the transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$ relative to the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solⁿ:- The ordered standard basis of $\mathbb{R}^2$ is

$B = \{(1, 0), (0, 1)\}$ and that of $\mathbb{R}^3$ is

$B' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

$$\therefore T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2, 5x_1 - 6x_2)$$

$$T(1, 0) = (3, 2, 5) \quad \& \quad T(0, 1) = (-1, 4, -6)$$

$$T(1, 0) = (3, 2, 5) = 3(100) + 2(010) + 5(001)$$

$$T(0, 1) = (-1, 4, -6) = -1(100) + 4(010) + (-6)(001)$$

Hence, matrix of the transformation is $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

2) Find the matrix representing the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by $T(x, y, z) = (x + y + z, 2x + z, 2y - z, 6y)$ relative to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

Solⁿ:- We know that ordered standard basis of $\mathbb{R}^3$ is

$$B = \{(1,0,0), (0,1,0), (0,0,1)\} \text{ and}$$

$$\mathbb{R}^4 \text{ is } B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

$$\text{Since } T(x,y,z) = (x+y+z, 2x+z, 2y-z, 6y)$$

$$T(1,0,0) = (1, 2, 0, 0)$$

$$T(0,1,0) = (1, 0, 2, 6)$$

$$T(0,0,1) = (1, 1, -1, 0)$$

$$\text{Let } e_1 = (1,0,0,0) \quad e_2 = (0,1,0,0) \quad e_3 = (0,0,1,0) \quad e_4 = (0,0,0,1)$$

$$\begin{aligned} \therefore T(1,0,0) &= (1, 2, 0, 0) = 1e_1 + 2e_2 + 0e_3 + 0e_4 \\ T(0,1,0) &= (1, 0, 2, 6) = 1e_1 + 0e_2 + 2e_3 + 6e_4 \\ T(0,0,1) &= (1, 1, -1, 0) = 1e_1 + 1e_2 - 1e_3 + 0e_4 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \rightarrow (1)$$

Matrix of T relative to B and B' is transpose of matrix of coefficients in above system of eqⁿ (1)

$$\text{i.e., } [T: B, B'] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

3) Find the linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ whose matrix is

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}, \text{ relative to the ordered basis}$$

$$B = \{(1,1), (0,2)\} \text{ and}$$

$$B' = \{(0,1,1), (1,0,1), (1,1,0)\} \text{ for } \mathbb{R}^3.$$

$$\text{Solⁿ:- Here } [T: B, B'] = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$$

$T(1,1)$ = Linear combination of vectors of B' using scalars of first column of $[T: B, B']$

$$= 1(0,1,1) - 2(1,0,1) + 0(1,1,0) = (-2, 1, -1)$$

$$\text{||}^4 T(0,2) = -1(0,1,1) + 3(1,0,1) + 1(1,1,0) = (4, 0, 2)$$

Let $(x, y) \in \mathbb{R}^2$ be arbitrary.

$$(x, y) = a(1, 1) + b(0, 2)$$

To find a and b :-

$$(x, y) = (a, a + 2b) \Rightarrow a = x$$

$$\text{and } a + 2b = y \Rightarrow b = \frac{y - x}{2}$$

Using these values,

$$(x, y) = x(1, 1) + \frac{y - x}{2}(0, 2)$$

Applying T on both sides,

$$T(x, y) = x T(1, 1) + \left(\frac{y - x}{2}\right) T(0, 2)$$

$$= x(-2, 1, -1) + \left(\frac{y - x}{2}\right)(4, 0, 2)$$

$$= \left[-2x + \left(\frac{y - x}{2}\right)4, x + 0, -x + \left(\frac{y - x}{2}\right)2\right]$$

$= (2y - 4x, x, y - 2x)$ which is the required transformation.

Change of Coordinates or change of Basis Matrix or Transition Matrix :-

Let $B_1 = \{u_1, u_2, \dots, u_n\}$ be a basis of n -dimensional vector space V and $B_2 = \{v_1, v_2, \dots, v_n\}$ be another basis of V . Then, each element in B_2 can be expressed as a linear combination of the vectors in basis B_1 .

$$\begin{aligned} \text{Let } v_1 &= a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n \\ v_2 &= a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n \\ &\vdots \\ v_n &= a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n \end{aligned}$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Then the transpose of the matrix of coefficients in above system of equations is said to be the transition matrix or change of basis matrix from the old basis B_1 to the new basis B_2 . It is denoted by P and written as

$$P = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

Since the vectors in B_2 are linearly independent, so the matrix P is invertible. Its inverse P^{-1} is the transition matrix from the basis B_2 to the basis B_1 .

Problems:-

1) Consider the following basis of $\mathbb{R}^2$:

$$\mathcal{E} = \{e_1, e_2\} = \{(1,0), (0,1)\} \text{ and } S = \{u_1, u_2\} = \{(1,3), (1,4)\}$$

(a) Find the change-of-basis matrix P from the usual basis $\mathcal{E}$ to S .

(b) Find the change-of-basis matrix Q from S back to $\mathcal{E}$.

(c) Find the coordinate vector $[v]$ of $v = (5, -3)$ relative to S .

(a) Since E is the usual basis of $\mathbb{R}^2$,

$$\therefore P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

(b) Find P^{-1} , $P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

(c) Let $[v]_S = P^{-1}[v]_E$.

$$\text{We have } [v]_E = [5, -3]^T$$

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}.$$

2) The vectors $u_1 = (1, 2, 0)$, $u_2 = (1, 3, 2)$, $u_3 = (0, 1, 3)$ form a basis S of $\mathbb{R}^3$. Find:

a) The change of basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ to the basis S .

b) The change of basis matrix Q from the above basis S back to the usual basis E of $\mathbb{R}^3$.

Soln:-

(a) Since E is a usual basis of $\mathbb{R}^3$,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find P^{-1} , $P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$

3) Find the co-ordinates of vector $(1,1,1)$ relative to basis $(1,1,2), (2,2,1), (1,2,2)$.

Solⁿ:- Let $(1,1,1) = a(1,1,2) + b(2,2,1) + c(1,2,2)$

$$a + 2b + c = 1 \rightarrow (1)$$

$$a + 2b + 2c = 1 \rightarrow (2)$$

$$2a + b + 2c = 1 \rightarrow (3)$$

subtracting (1) from (2), we have $c = 0$

Putting $c = 0$ in (2) and (3), we get

$$a + 2b = 1 \rightarrow (4)$$

$$2a + b = 1 \rightarrow (5)$$

Solving (4) and (5), we get $a = b = \frac{1}{3}$.

$\therefore$ Coordinates of $(1,1,1)$ relative to given basis are $(\frac{1}{3}, \frac{1}{3}, 0)$.

4) Find the co-ordinates of the following vectors relative to the basis $\gamma_1 = (1,1,2), \gamma_2 = (2,2,1), \gamma_3 = (1,2,2)$.

(i) $(1,0,1)$ solⁿ:- $(\frac{4}{3}, \frac{1}{3}, -1)$

(ii) $(1,1,0)$ solⁿ:- $(-\frac{1}{3}, \frac{2}{3}, 0)$

Rank and nullity of a linear Operator:-

Rank of T:-

Dimension of the $\text{Im}(T)$ is known as rank of T .

Nullity of T:-

Dimension of the $\text{Ker}(T)$ is known as nullity of T .

Rank-nullity Theorem:-

Let V and W be vector spaces over the field F and let T be a linear transformation from V to W . If V is a finite dimensional then

$$\text{rank}(T) + \text{nullity of } (T) = \dim(V).$$

Problems:-

1) For the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$, find a basis and dimension of its range space and its null space. Also verify, that $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^2$.

Solⁿ:- i) To find null space of T and its dimension:

By defⁿ, null space = $N(T) = \{u \in \mathbb{R}^2 : T(u) = 0 \in \mathbb{R}^3\}$

Let $u = (x_1, x_2) \in \mathbb{R}^2$ be an arbitrary element of null space.

$$T(u) = 0$$

$$T(x_1, x_2) = 0$$

$$(x_1 - x_2, x_2 - x_1, -x_1) = 0$$

$$\begin{aligned}\Rightarrow x_1 - x_2 &= 0 \\ x_2 - x_1 &= 0 \\ -x_1 &= 0\end{aligned}$$

On solving the above equations, We get

$$x_1 = 0, x_2 = 0.$$

Thus, the only member of null space is a zero vector $\in \mathbb{R}^3$
i.e. $N(T) = \{0\}$.

$$\therefore \text{Nullity of } T = \dim(N(T)) = 0.$$

ii) To find range space of T and its dimension:

The range space consists of ordered triples

$(x_1 - x_2, x_2 - x_1, -x_1)$ for $(x_1, x_2) \in \mathbb{R}^2$ such that

$$T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1) \rightarrow (1)$$

Let v be any element of $R(T)$.

Thus there exist $(x_1, x_2) \in \mathbb{R}^2$ such that $v = T(x_1, x_2)$

$$\Rightarrow (x_1, x_2) = x_1(1, 0) + x_2(0, 1)$$

$$= x_1 e_1 + x_2 e_2 \text{ where } e_1 = (1, 0), e_2 = (0, 1)$$

$$T(x_1, x_2) = T(x_1 e_1 + x_2 e_2)$$

$$= x_1 T(e_1) + x_2 T(e_2) \rightarrow (2)$$

$$T(e_1) = T(1, 0) = (1 - 0, 0 - 1, -1) = (1, -1, -1)$$

$$T(e_2) = T(0, 1) = (0 - 1, 1 - 0, 0) = (-1, 1, 0)$$

Putting the values of $T(e_1), T(e_2)$ in (2)

$$T(x_1, x_2) = x_1(1, -1, -1) + x_2(-1, 1, 0)$$

$$v = x_1(1, -1, -1) + x_2(-1, 1, 0) \rightarrow (3)$$

Since $v \in R(T)$ is arbitrary,

(iii) Let $a(1, -1, -1) + b(-1, 1, 0) = 0$ for $a, b \in F$

$$\Rightarrow (a-b, -a+b, -a) = (0, 0, 0)$$

$$\Rightarrow a-b=0, b-a=0, -a=0$$

$$\Rightarrow a=b=0.$$

Thus $\{(1, -1, -1), (-1, 1, 0)\}$ is linearly independent, $\therefore$ It is a basis set of $R(T)$.

$$\therefore \text{Dimension of } R(T) = 2$$

$$\text{Also dimension of } R^2 = 2$$

$$\therefore \text{Rank}(T) + \text{nullity}(T) = 2 + 0 = 2 = \dim R^2.$$

2) Find a linear transformation $T: R^3 \rightarrow R^3$ whose range space is spanned by the vectors $(1, 2, 3), (4, 5, 6)$

Solⁿ: - We know that $\{e_1, e_2, e_3\}$ is a standard basis of R^3 , where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

Since R^3 is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of R^3 , there exist a unique linear transformation T such that

$$T(e_1) = (1, 2, 3)$$

$$T(e_2) = (4, 5, 6)$$

$$T(e_3) = (0, 0, 0) \quad (\text{Assuming } T(0, 0, 1) = (0, 0, 0))$$

Also $R(T)$ is spanned by $\{(1, 2, 3), (4, 5, 6)\}$

i.e by $\{(1, 2, 3), (4, 5, 6), (0, 0, 0)\}$ or by $\{T(e_1), T(e_2), T(e_3)\}$

Now, for each $(x_1, x_2, x_3) \in R^3$,

$$\begin{aligned} (x_1, x_2, x_3) &= x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1) \\ &= x_1 e_1 + x_2 e_2 + x_3 e_3 \end{aligned}$$

$$\begin{aligned}
 T(x_1, x_2, x_3) &= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) \\
 &= x_1 (1, 2, 3) + x_2 (4, 5, 6) + x_3 (0, 0, 0) \\
 &= (x_1 + 4x_2, 2x_1 + 5x_2, 3x_1 + 6x_2)
 \end{aligned}$$

Which is the required linear transformation.

3) Verify the Rank-nullity theorem for the $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

Soln:- To find the basis of nullity of T

Let $v = (x, y, z) \in \mathbb{R}^3$ such that

$$\text{Nullity of } T = \{v \in V \mid T(v) = 0\}$$

$$T(x, y, z) = 0.$$

$$(x + 2y - z, y + z, x + y - 2z) = (0, 0, 0).$$

$$\Rightarrow x + 2y - z = 0 \Rightarrow x = 3z$$

$$y + z = 0 \Rightarrow y = -z.$$

$$x + y - 2z = 0$$

On solving above equations we get

$$x = 3z, \quad y = -z$$

$$\therefore \{(x, y, z)\} = \{3z, -z, z\} = (z\{3, -1, 1\})$$

Thus $\{(3, -1, 1)\}$ is a basis of nullity of T

$$\& \text{ Nullity}(T) = 1.$$

To find the basis of Range of T .

As $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ generates $\mathbb{R}^3$

$\Rightarrow \{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\}$ generates range of T

$\Rightarrow \{(1, 0, 1), (2, 1, 1), (-1, 1, -2)\}$ generates range of T

To find the basis of range,
Consider a matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
 $R_3 \rightarrow R_3 + R_2$ $R_3 \rightarrow R_3 - R_2$

Thus $\{(1, 0, 1), (0, 1, -1)\}$ form a basis of Range of T & $\dim(\text{Range of } T) = 2$.

$$\begin{aligned} \therefore \text{Rank}(T) + \text{Nullity}(T) &= 2 + 1 \\ &= 3 = \dim(R^3). \end{aligned}$$

Hence Rank-Nullity theorem verified.

4) Verify the Rank-Nullity theorem for the
 $T: R^3 \rightarrow R^3$ defined by $T(x, y, z) = (x+2y, y-z, x+2z)$.

Inner Product Spaces and Orthogonality.

Inner Product Space :-

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

- (i) Linear Property: $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$
- (ii) Symmetric Property: $\langle u, v \rangle = \langle v, u \rangle$
- (iii) Positive Definite Property: $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ if and only if $u = 0$.

Norm of a vector :

An inner product, $\langle u, u \rangle$ is nonnegative for any vector u .

$$\|u\| = \sqrt{\langle u, u \rangle} \quad \text{or} \quad \|u\|^2 = \langle u, u \rangle$$

This non negative number is called the norm or length of u .

* Every non zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|} v$. which is a positive multiple of v .

This process is called normalizing v .

Orthogonality:-

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v \rangle = 0.$$

Problems:-

1) Consider vectors $u = (1, 2, 4)$, $v = (2, -3, 5)$, $w = (4, 2, -3)$ in $\mathbb{R}^3$. Find a) $\langle u, v \rangle$ b) $\langle u, w \rangle$ c) $\langle v, w \rangle$ d) $\langle u+v, w \rangle$

e) $\|u\|$ f) $\|v\|$

a) $\langle u, v \rangle = 2 - 6 + 20 = 16$

b) $\langle u, w \rangle = 4 + 4 - 12 = -4$

c) $\langle v, w \rangle = 8 - 6 - 15 = -13$

d) $\langle u+v, w \rangle = (3, -1, 9) \cdot (4, 2, -3) = 12 - 2 - 27 = -17$

e) $\|u\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$

f) $\|v\| = \sqrt{4 + 9 + 25} = \sqrt{38}$

2) Consider the vectors $u = (1, 5)$ and $v = (3, 4)$ in $\mathbb{R}^2$. Find : a) $\langle u, v \rangle$ with respect to the usual inner product in $\mathbb{R}^2$.

b) $\|v\|$ using the inner product in $\mathbb{R}^2$.

3) Consider the following polynomials in $P(t)$ and inner product:

$$f(t) = t+2, \quad g(t) = 3t-2, \quad h(t) = t^2-2t-3 \text{ and}$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$ (b) find $\|f\|$ and $\|g\|$

(c) normalize f and g .

Solⁿ: (a) $\langle f, g \rangle = \int_0^1 (t+2)(3t-2) dt = \int_0^1 (3t^2 + 4t - 4) dt$

$$\langle f, g \rangle = t^3 + 2t^2 - 4t \Big|_0^1 = -1.$$

$$\langle f, h \rangle = \int_0^1 (t+2)(t^2-2t-3) dt = \left[\frac{t^4}{4} - \frac{7t^3}{2} + 6t^2 \right]_0^1 = -\frac{37}{4}$$

(b) $\langle f, f \rangle = \int_0^1 (t+2)(t+2) dt = \frac{19}{3}; \|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}$

$$\langle g, g \rangle = \int_0^1 (3t-2)(3t-2) dt = 1; \|g\| = \sqrt{1} = 1$$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}} (t+2)$$

$$\hat{g} = \frac{1}{\|g\|} g = 3t-2.$$

4) Let $M = M_{2,3}$ with inner product. $\langle A, B \rangle = \text{tr} \langle B^T A \rangle$

and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$

Find (a) $\langle A, B \rangle$, $\langle A, C \rangle$, $\langle B, C \rangle$

(b) $\langle 2A+3B, 4C \rangle$ (c) $\|A\|$ and $\|B\|$

$$(a) \quad \langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24 = 119$$

$$\langle A, C \rangle = 27 - 40 + 14 + 6 + 0 - 16 = -9$$

$$\langle B, C \rangle = 3 - 10 + 6 + 4 + 0 - 24 = -21$$

$$(b) \quad 2A + 3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix} \quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$$

$$\langle 2A + 3B, 4C \rangle = 252 - 440 + 96 + 0 - 416 = -324$$

$$(c) \quad \|A\|^2 = \langle A, A \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2, \text{ the sum of the}$$

squares of all the elements of A.

$$\|A\|^2 = \langle A, A \rangle = 9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$$

$$\|B\|^2 = \langle B, B \rangle = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$$

4) Verify the vectors $u = (1, 1, 1)$, $v = (1, 2, -3)$ & $w = (1, -4, 3)$ in R^3 are orthogonal or not.

Soln: - $\langle u, v \rangle = 1 + 2 - 3 = 0$, $\langle u, w \rangle = 1 - 4 + 3 = 0$, $\langle v, w \rangle = 1 - 8 - 9 = -16$

Thus u is orthogonal to v and w ,
 v & w are not orthogonal.