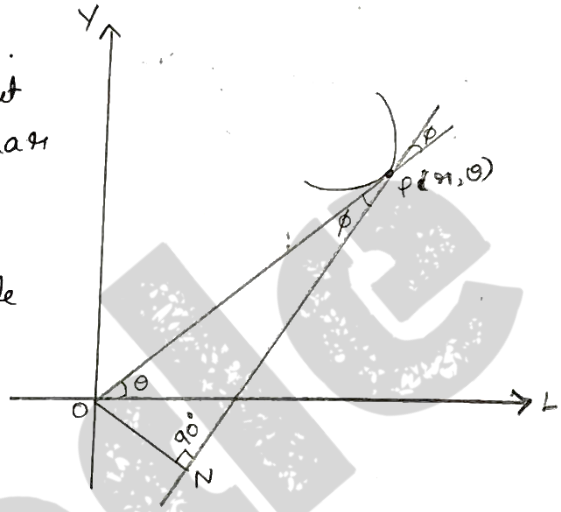
Module - 1

Q.01

a) with usual notation prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

Ans: let O be the pole and OL be the initial line. let $P(r,\theta)$ be any point on the curve so that $OP = r$, the radius vector and $\angle LOP = \theta$, the polar angle.



Draw $ON = p$ (say) a $\perp^r$ from the pole on the tangent at P. let ϕ be the angle b/n made by the radius vector with the tangent.

From the fig, $O\hat{N}P = 90°$

From the right angled $\triangle^{le}$ ONP,

we have $\sin\phi = \frac{opp}{hyp} = \frac{p}{r}$.

$$\boxed{p = r\sin\phi} \quad \text{---} \ (1)$$

This is the expression for length of the $\perp^r$ p to express p in terms of θ.

Squaring on b.s of eq (1)

$$p^2 = r^2 \sin^2\phi$$

taking reciprocal on b.s.

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \times \operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} \times [1 + \cot^2\phi]$$

W.KT $\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Hence the proof

b. Find the angle between curves $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$

Ans: consider $r = \frac{a}{1+\cos\theta}$

Take log on both sides

$\log r = \log a - \log(1+\cos\theta)$

diff w.r.t θ.

$\frac{d}{d\theta}(\log r) = \frac{d}{d\theta}(\log a) - \frac{d}{d\theta}(\log(1+\cos\theta))\frac{d}{d\theta}(\cos\theta)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{0-(-\sin\theta)}{1+\cos\theta}$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$

$\boxed{\cot\phi_1 = \frac{\sin\theta}{1+\cos\theta}}$

consider $r = \frac{b}{1-\cos\theta}$.

Take log on both sides

$\log r = \log b - \log(1-\cos\theta)$

diff w.r.t θ.

$\frac{d}{d\theta}(\log r) = \frac{d}{d\theta}(\log b) - \frac{d}{d\theta}(\log(1-\cos\theta))\frac{d}{d\theta}(1) - \frac{d}{d\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{0-(\sin\theta)}{1-\cos\theta}$

$\boxed{\cot\phi_2 = \frac{-\sin\theta}{1-\cos\theta}}$

$\therefore |\phi_1 - \phi_2|$ or $\cot\phi_1 \times \cot\phi_2 = -1$

$\therefore \frac{\sin\theta}{1+\cos\theta} \times \frac{-\sin\theta}{1-\cos\theta} = -1$

$\because (a+b)(a-b) = a^2 - b^2$

$\therefore 1^2 - \cos^2\theta = \sin^2\theta.$

$\therefore \frac{-\sin^2\theta}{1^2-\cos^2\theta} = -1$

$\therefore \frac{-\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}} = -1$

$\therefore -1 = -1$

then $|\phi_1 - \phi_2| = \pi/2$

c) Find the radius of curvature of the curve $y = x^3(x-a)$ at the point $(a, 0)$

Ans:- consider $y = x^3(x-a)$

$y = x^4 - ax^3$

diff w.r.t 'x'

$y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^4) - a\frac{d}{dx}(x^3)$

$y_1 = 4x^3 - 3ax^2$

$\boxed{y_1 = 4x^3 - 3ax^2}$

again diff y_1 w.r.t 'x'

$y_2 = \frac{d^2y}{dx^2} = 4\frac{d}{dx}(x^3) - 3a\frac{d}{dx}(x^2)$

$\boxed{y_2 = 12x^2 - 6ax}$

at $(a,0)$

$y_1 = 4(a)^3 - 3a(a)^2$

$= 4a^3 - 3a^3$

$\boxed{y_1 = a^6}$

at $(a,0)$

$y_2 = 12(a)^2 - 6a(a)$

$y_2 = 12a^2 - 6a^2$

$\boxed{y_2 = 6a^2}$

$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$\rho = \frac{(1+(a^3)^2)^{3/2}}{6a^2}$

$|\rho| = \frac{(1+a^6)^{3/2}}{6a^2}$

or $|\rho| = \frac{\sqrt[3]{1+a^6}}{6a^2}$

Module-01

Q.02

a) Show that the curves $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ cuts each other orthogonally.

Ans:- Consider $r = a(1+\cos\theta)$

Take log on both sides

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t 'θ'

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{(-\sin\theta)}{1+\cos\theta}$

$\boxed{\cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}}$

Consider $r = a(1-\cos\theta)$

Take log on both sides

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t θ.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

$\boxed{\cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}}$

Then $\cot\phi_1 \times \cot\phi_2 = -1$

$\frac{-\sin\theta}{1+\cos\theta} \times \frac{\sin\theta}{1-\cos\theta} = -1$

$\therefore (a+b)(a-b) = a^2 - b^2$

$\therefore \sin^2\theta = 1 - \cos^2\theta$.

$\frac{-\sin^2\theta}{1^2-\cos^2\theta} = -1$

$\frac{-\sin^2\theta}{\sin^2\theta} = -1$

$-1 = -1$

therefore $|\phi_1 - \phi_2| = \pi/2$.

Therefore, the curves which cuts each other orthogonally.

Consider, $x_1 = \frac{2xy}{-a^2 \cdot y^2}$

$(a^2+y^2)x_1 = 2xy$

again diff w.r.t y.

$a^2+y^2 \frac{d}{dy}(x_1) + x_1\left(\frac{d}{dx}(a^2) + \frac{d}{dx}(y^2)\right) = -2\left[x\frac{dy}{dy} + y\frac{dx}{dy}\right]$

$(a^2+y^2)x_2 + x_1(0+2y) = -2[x + yx_1]$

at $(a,0)$

$$(a^2+y^2)x_2 + x_1(2y) = -2[x+yx_1]$$

$$(a^2+0)x_2 + 0(2(0)) = -2[a+(0)(0)]$$

$$a^2x_2 = -2a$$

$$x_2 = \frac{-2\cancel{a}}{a\cancel{^2}}$$

$$\boxed{x_2 = \frac{-2}{a}}$$

R.O.C $\quad \rho = \dfrac{(1+(x_1)^2)^{3/2}}{x^2}$

$$\rho = \frac{(1+0)^{3/2}}{-2/a}$$

$$\rho = \frac{1}{-2/a}$$

$$\rho = \frac{a}{-2}$$

$$|\rho| = \left|\frac{a}{-2}\right|$$

$$\boxed{|\rho| = a/2}$$

b> Find the pedal equation of the curve $r(1-\cos\theta) = 2a$.

Ans:- Consider the equation $r(1-\cos\theta) = 2a$.

Take log on both sides.

$\log r + \log(1-\cos\theta) = \log 2a$

diff w.r.t θ.

$$\frac{d}{d\theta}(\log r) + \frac{d}{d\theta}(1-\cos\theta)\frac{d}{d\theta}(1) - \frac{d}{d\theta}(\cos\theta) = \frac{d}{d\theta}(\log 2a)$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0.$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\cot\phi = \frac{-\sin\theta}{1-\cos\theta}$$

From the pedal equation $\frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1^2+\cos^2\theta-2\cos\theta+\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+1-2\cos\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2-2\cos\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{\cancel{1-\cos\theta}}{(1-\cos\theta)^{\cancel{2}}}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{1}{1-\cos\theta}\right] \rightarrow (1)$$

To eliminate θ,

From given $r(1-\cos\theta) = 2a$

$$\therefore \frac{r}{2a} = \frac{1}{1-\cos\theta}.$$

then eq ① becomes $\frac{1}{p^2} = \frac{\cancel{r}}{r\cancel{r}} \cdot \frac{r}{\cancel{r}a}$

$\Rightarrow \frac{1}{p^2} = \frac{1}{r} \cdot \frac{1}{a}$

$p^2 = ra$

$\boxed{p = \sqrt{ra}}$

c. Find the radius of curvature for the curve $y^2 = \frac{a^2(a-x)}{x}$, where the curve meets x axis.

Ans:- The given curve $y^2 = \frac{a^2(a-x)}{x}$, then curve meet at point $(a,0)$

$xy^2 = a^3 - a^2x$

diff w.r.t 'x'

$x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) = \frac{d}{dx}(a^3) - a^2\frac{d}{dx}(x)$

$x \cdot 2y\frac{dy}{dx} + y^2 = -a^2.$

$x \cdot 2y\frac{dy}{dx} + y^2 = -a^2$

$x \cdot 2y \cdot y_1 + y^2 = -a^2$

$\boxed{y_1 = \frac{-a^2-y^2}{2xy}}$

at $(a,0)$

$y_1 = \frac{-a^2-y^2}{2xy}$

$y_1 = \frac{-a^2-0}{+2(a)(0)}$

$\boxed{y_1 = \infty}$

Since $y_1 = 0$ at $(a,0)$ we consider.

$x_1 = \frac{dx}{dy} = \frac{2xy}{-a^2-y^2}$

at $(a,0)$

$x_1 = \frac{2xy}{-a^2-y^2}$

$x_1 = \frac{2(a)(0)}{-a^2-0}$

$\boxed{x_1 = 0}$

Module - 02

Q no:- 3

Q.03 a) Expand $\log(1+\sin x)$ up to the term containing x^4 using maclaurin's series

Ans:- $y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \ldots \rightarrow (1)$.

let $y = \log(1+\sin x)$; $y(0) = \log(1+\sin 0) = \log 1 = \boxed{0}$

$y_1(x) = \frac{1}{1+\sin x}(\cos x)$; $y_1(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1} = \boxed{1}$

$y_1(1+\sin x) = \cos x$

diff w.r.t 'x'

$y_1(\cos x) + (1+\sin x)y_2 = -\sin x$

$(1)(1) + (1+0)y_2 = 0$

$y_2(0) = \boxed{-1}$

consider $y_1(\cos x) + (1+\sin x)y_2 = -\sin x$

diff $[y_1(-\sin x) + \cos x \cdot y_2] + [(1+\sin x)y_3 + y_2(\cos x)] = -\cos x$

$(1(0) + 1(-1)) + ((1)y_3 + (-1)(1)) = -1$

$-1 + y_3 - 1 = -1$

$y_3 = -1+2$

$\boxed{y_3 = 1}$

consider $[y_1(-\sin x) + (\cos x)y_2] + [(1+\sin x)y_3 + y_2(\cos x)] = -\cos x$

diff $[y_1(-\cos x) + (-\cancel{\sin x}^{0}\, y_2) + (\cos x\, y_3 + y_2 \cdot -\cancel{\sin x}^{0})] +$
$[(1+\cancel{\sin x}^{0})y_4 - (\cos x)) + (y_2 \cdot (-\sin x) + (\cos x)(y_3))] = \cancel{\sin x}^{0}$

$[(1)(-1) + (1)(1)] + ((-1)(y_4) + (1)(1) + (1)(1)] = 0$

$\cancel{-1} \cancel{+1} + y_4 + 1 + 1 = 0$

$\boxed{y_4 = -2}$

put in eq^{tn} (1)

$\therefore y(x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-2)$

$\therefore y(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$

b. if $u = \log(\tan x + \tan y + \tan z)$ show that.

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Ans:- Consider $u = \log(\tan x + \tan y + \tan z)$

$$\boxed{e^u = \tan x + \tan y + \tan z}$$

① partial diff w.r.t x.

$$\frac{\partial}{\partial x}(e^u) = \frac{\partial}{\partial x}(\tan x)$$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{e^u}$$

multiply $\sin 2x$ on both side

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{e^u}$$

$$\boxed{\sin 2x \frac{\partial u}{\partial x} = \frac{2\tan x}{e^u}} \rightarrow ①$$

② partial diff w.r.t y.

$$\frac{\partial}{\partial y}(e^u) = \frac{\partial}{\partial y}(\tan y)$$

$$\frac{\partial u}{\partial y} e^u = \sec^2 y$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{e^u}$$

multiply by $\sin 2y$ on both side.

$$\sin 2y \frac{\partial u}{\partial y} = \frac{\sin 2y \sec^2 y}{e^u}$$

$$\boxed{\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{e^u}} \rightarrow ②$$

③ $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2}\frac{\partial u}{\partial q} + \frac{1}{x}\frac{\partial u}{\partial r}$$

multiply b.s by z

$$\boxed{z \cdot \frac{\partial u}{\partial z} = -\frac{yz}{z^2}\frac{\partial u}{\partial q} + \frac{z}{x}\frac{\partial u}{\partial r}} \longrightarrow ③$$

By adding eq ① ② and ③

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}}\frac{\partial u}{\partial p} - \cancel{\frac{z}{x}}\frac{\partial u}{\partial r} + \left(\cancel{\frac{-x}{y}}\right)\frac{\partial u}{\partial p} + \cancel{\frac{y}{z}}\frac{\partial u}{\partial q} - \cancel{\frac{y}{z}}\frac{\partial u}{\partial q} + \cancel{\frac{z}{x}}\frac{\partial u}{\partial z}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

c. If $x = r\sin\theta\cos\psi$ $y = r\sin\theta\sin\psi$ $z = r\cos\theta$. Find the value of

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\psi)} = ?$$

Sol^n:-

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\psi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \psi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \psi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \psi} \end{vmatrix}$$

Consider :-

$x = r\sin\theta\cos\psi$ $y = r\sin\theta\sin\psi$ $z = r\cos\theta.$

$\frac{\partial x}{\partial r} = \sin\theta\cos\psi$ $\frac{\partial y}{\partial r} = \sin\theta\sin\psi$ $\frac{\partial z}{\partial r} = \cos\theta.$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\psi$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\psi$ $\frac{\partial z}{\partial \theta} = -r\sin\theta.$

$\frac{\partial x}{\partial \psi} = -r\sin\theta\sin\psi$ $\frac{\partial y}{\partial \theta} = r\sin\theta\cos\psi$ $\frac{\partial z}{\partial \psi} = 0.$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\psi)} = \begin{vmatrix} \sin\theta\cos\psi & r\cos\theta\cos\psi & -r\sin\theta\sin\psi \\ \sin\theta\sin\psi & r\cos\theta\sin\psi & r\sin\theta\cos\psi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$J = \sin\theta\cos\psi\,[0 + r^2\sin^2\theta\cos^2\psi] - r\cos\theta\cos\psi\,[0 - r\sin\theta\cos\theta\cos\psi] - r\sin\theta\sin\psi\,.$$
$$[-r\sin^2\theta\sin\psi - r^2\cos^2\theta\sin\psi]$$

$$J = r^2\sin^3\theta\cos^2\psi + r^2\sin\theta\cos^2\theta\cos\psi + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin^2\cos^2\psi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin\theta\cos^2\psi + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin\theta\,[\cos^2\psi + \sin^2\psi]$$

$$J = \underline{\underline{r^2\sin\theta}}$$

Q.04) a) Evaluate $\lim_{x\to 0}\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$

$$\log_e k = \lim_{x\to 0}\log\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$$

$$= \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log\left(\frac{a^x + b^x + c^x}{3}\right)}{x}\quad\left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0}\frac{\frac{1}{\frac{a^x + b^x + c^x}{\cancel{3}}}\cdot\frac{1}{\cancel{3}}\left[a^x\log a + b^x\log b + c^x\log c\right]}{1}$$

$$= \lim_{x\to 0}\frac{a^x\log a + b^x\log b + c^x\log c}{a^x + b^x + c^x}$$

$$= \frac{\log a + \log b + \log c}{3}$$

$$= \frac{1}{3}\log(abc)$$

$$= \log(abc)^{1/3}$$

$$k = (abc)^{1/3}$$

b) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

$p = \frac{x}{y} \quad q = \frac{y}{z} \quad r = \frac{z}{x}$

$u \rightarrow (p, q, r) \rightarrow \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

$u \rightarrow \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}\cdot(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right) \quad \text{x}^{ly} \text{ by } x$$

$$x u_x = \frac{\partial u}{\partial p}\left(\frac{x}{y}\right) + \frac{\partial u}{\partial r}\left(\frac{-z}{x}\right) \quad -(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial q} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$u_y = \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0) \quad x^{ly} \text{ by } y.$$

$$y u_y = \frac{\partial u}{\partial p}\left(\frac{-x}{y}\right) + \frac{\partial u}{\partial q}\left(\frac{y}{z}\right) \quad -(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$u_z = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right) \quad x^{ly} \text{ by } z$$

$$z u_z = -\frac{y}{z}\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}\left(\frac{z}{x}\right) \rightarrow (3)$$

Adding (1) (2) and (3)

$$x u_x + y u_y + z u_z = 0.$$

c) If $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$ find the value of $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta . \cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta . \sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta . \cos\phi\,(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,[0 - r\sin\theta\cos\phi . \cos\theta] .$
$r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$

$= r^2\sin^3\theta . \cos^2\phi + r^2\sin\theta\cos^2\phi - (r\sin\theta\sin\phi)(-r\sin\phi)(\sin^2\theta + \cos^2\theta)$

$= r^2\sin^3\theta\,\cos^2\phi + r^2\sin\theta\cos^2\theta\,\cos^2\phi + r^2\sin\theta\,\sin^2\phi$

$= r^2\sin\theta\,\cos^2\phi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\sin^2\phi$

$= r^2\sin\theta\,\cos^2\phi + r^2\sin\theta\,\sin^2\phi$

$= r^2\sin\theta\,[\cos^2\phi + \sin^2\phi]$

$= r^2\sin\theta$

Q.5 Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

Ans: Given the diff equation is in the form of

$\frac{dy}{dx} + py = xy^n$

Divide given equation by y^2

$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{xy^2} = \frac{xy^2}{y^2}$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = x \rightarrow ①$

Substitute $\boxed{\frac{1}{y} = t}$

diff w.r.t x.

$\frac{dt}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$

$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx} \rightarrow ②$

Sub eq ② in eq ①

$-\frac{dt}{dx} + \frac{1}{x}\cdot t = x$

Above equation multiply by -1

$\frac{dt}{dx} - \frac{1}{x}t = -x$

$P = -\frac{1}{x}$ $\quad Q = -x$

$IF = e^{\int P dx}$

$= e^{-\int \frac{1}{x} dx}$

$IF = e^{-\log x}$

$IF = e^{\log x^{-1}}$

$IF = x^{-1}$

$\boxed{IF = \frac{1}{x}}$

Solution: $t[IF] = \int Q[IF]\,dx + c$

$$t\cdot\frac{1}{x} = \int -x\cdot\frac{1}{x}\,dx + c$$

$$t\cdot\frac{1}{x} = -\int 1\,dx + c$$

$$\frac{1}{y}\frac{1}{x} = -x + c$$

$$\boxed{\frac{1}{xy} = -x + c}$$

b) Find the orthogonal trajectories of $r = a(1+\cos\theta)$ where a is parameter

Soln: $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan\theta/2$$

$$\frac{1}{\tan\theta/2}\,d\theta = \frac{1}{r}\,dr.$$

$$\cot\theta/2\,d\theta = \frac{1}{r}\,dr$$

Integrate on b.s

$$\int\frac{1}{r}\,dr = \int\cot\theta/2\,d\theta + c$$

$$\log r = \frac{\log(\sin\theta/2)}{1/2} + c$$

$$\log r = 2\log(\sin\theta/2) + \log k$$

$$\log r = \log\sin^2\theta/2 + \log k$$

$$\log r = \log(k\sin^2\theta/2)$$

$$r = k\sin^2\theta/2.$$

$$r = k\left[\frac{1-\cos 2\theta}{2}\right] \qquad \frac{k}{2} = c_1$$

$$\boxed{r = c_1(1-\cos 2\theta)}$$

Q) Solve $p^2 + 2py\cot x - y^2 = 0$

Ans) Here Equation in form of $ax^2 + bx + c = 0$

$a = 1 \quad b = 2y\cot x \quad c = -y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\,\mathrm{cosec}\,x}{2}$$

$$p = -y\cot x \pm y\,\mathrm{cosec}\,x$$

$p = -y\cot x + y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = y(\mathrm{cosec}\,x - \cot x)$

$\int \frac{1}{y}\,dy = \int \mathrm{cosec}\,x\,dx - \int \cot x\,dx$

$\log y = \log(\mathrm{cosec}\,x - \cot x) - \log \sin x + \log k$

$\log y = \log\left[\frac{k\,\mathrm{cosec}\,x - \cot x}{\sin x}\right]$

$y = k\,\mathrm{cosec}\,x - \cot x$

$y\sin x = -k(\mathrm{cosec}\,x - \cot x) = 0$

$p = -y\cot x - y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = -y(\cot x + \mathrm{cosec}\,x)$

$-\int \frac{1}{y}\,dy = \int \cot x\,dx + \int \mathrm{cosec}\,x\,dx$

$-\log y = \log(\sin x) + \log(\mathrm{cosec}\,x - \cot x) + \log k$

$\log y + \log(\sin x) + \log(\mathrm{cosec}\,x - \cot x) = -\log k$

$y\sin x(\mathrm{cosec}\,x - \cot x) + k = 0$

General solution:-

$y\sin x - k(\mathrm{cosec}\,x - \cot x) \cdot (y\sin x(\mathrm{cosec}\,x - \cot x) + k) = 0$

Q.06 a) Solve $y(2x+1)dx - x\,dy = 0$

$M = 2xy^2+y \qquad N = -x\,dy$

$\frac{\partial M}{\partial y} = 4xy+1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{-1(4xy+1)}{} = -1-4xy-1 \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$= -4xy-2$ close to M.

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2(\cancel{2xy+1})}{y(\cancel{2xy+1})}$

$F(x) = -2/y$

$IF = e^{\int b(x)}$

$\Rightarrow e^{-2\int \frac{1}{y}} = e^{\log y^{-2}}$

$\boxed{I_F = \frac{1}{y^2}}$

multiply $\frac{1}{y^2}$ on given equation

$(2x+\frac{1}{y^2})dx - \frac{x}{y^2}dy = 0$

$M = 2x+\frac{1}{y^2} \qquad N = -x.\frac{1}{y^2}dy$

$\frac{\partial M}{\partial y} = -2y^{-3} \qquad \frac{\partial N}{\partial x} = -2y^{-3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\int M dx + \int$ term of N free from x

$-2\int y^{-3} + \int 0\,d$

$\frac{-2y^{-2}}{\cancel{-2}} + c$

$y^{-2}+c$

$\frac{1}{y^2}+c$

b) Find the orthogonal trajectories of family $r^n \sin n\theta = a^n$

Ans: $r^n \sin n\theta = a^n$

take log on both side

$\log r^n + \log \sin n\theta = \log a^n$

$n \log r + \log \sin n\theta = n \log a$

diff w.r.t θ.

$n \cdot \frac{1}{r} \frac{dr}{d\theta} + \frac{\cos n\theta \cdot n}{\sin n\theta} = 0$

$\frac{1}{r} \frac{dr}{d\theta} + \cot n\theta = 0$

replace $\frac{dr}{d\theta}$ b g $-r^2 \frac{d\theta}{dr}$

$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) \cancel{+} \cot n\theta = 0$

$\cancel{-} r \frac{d\theta}{dr} = \cancel{-} \cot n\theta$

$\frac{1}{r} dr = \frac{1}{\cot n\theta} \cdot d\theta$

$\int \frac{1}{r} dr = \int \tan n\theta \, d\theta + c$

$\log r = \frac{\log (\sec n\theta)}{n} + c$

$n \log r = \log (\sec n\theta) + nc$

$n \log r = \log (\sec n\theta) + \log b$

$\log r^n = \log (b \sec n\theta)$

$r^n = b \sec n\theta$

$\boxed{r^n \cos n\theta = b}$

Q. Find the general solution of the equation $(px-y)(py+x)=2p$ by reducing into Clairaut's form by taking the substitution $X = x^2$, $Y = y^2$

Ans: Given $X = x^2 \quad Y = y^2$.

$$\frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

$$p = \frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$p = \frac{1}{\cancel{2}y}\,P.\,\cancel{2}x$$

$$p = \frac{x}{y}\,P$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}}\,.P.$$

Substitute in ①

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}\,.P.\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}\,P.\sqrt{Y} + \sqrt{X}\right) = \frac{2\sqrt{X}}{\sqrt{Y}}\,.P.$$

$$\left[\frac{XP}{\sqrt{Y}} - Y\right]\left[\sqrt{X}\,P + \sqrt{X}\right] = \frac{2\sqrt{X}}{\sqrt{Y}}\,.P.$$

$$\left[\frac{XP-Y}{\cancel{\sqrt{Y}}}\right]\cancel{\sqrt{X}}\,[P+1] = \frac{2\cancel{\sqrt{X}}}{\cancel{\sqrt{Y}}}\,.P$$

$$[XP-Y][P+1] = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$Y = PX - \frac{2P}{P+1}$$

This is in Clairaut's form

General solution $Y = cX - \dfrac{2c}{c+1}$

$$\boxed{y^2 = cx^2 - \frac{2c}{c+1}}$$

Module - 04

07) a) i) Find the remainder when 2^{23} is divided by 47.

Sol^n:- $2^8 = 256 = 21 \pmod{47}$

$(2^8)^2 = (21)^2 \pmod{47}$

$2^{16} = 441 \pmod{47}$

$2^{16} = 18 \pmod{47} \rightarrow ①$

consider $2^7 = 128 \equiv 34 \pmod{47}$ - ②.

eq ① x ②

$2^{16}.\,2^7 \equiv (18 \times 34) \pmod{47}$

$2^{23} \equiv 612 \pmod{47}$

$2^{23} = 1 \pmod{47}$

∴ 1 is the remainder when 2^{23} is divided by 47.

ii) Find the last digit in 7^{118}.

Sol^n:-

$$\begin{array}{r|l} 4)\,118\,(29 & \\ 8\downarrow & \\ \hline 38 & \\ 36 & \\ \hline 02 & \end{array}$$

$7^{4k} = 1 \pmod{10}$

$7^{4k+1} = 7 \pmod{10}$

$7^{4k+2} = 9 \pmod{10}$

$7^{4k+3} = 3 \pmod{10}$

$\therefore 7^{118} = 7^{4 \times 29 + 2}$

$7^{118} = 7^{4k+2} = 9 \pmod{10}$

∴ 9 is the last digit.

b) Find the solutions of the linear congruence $11x \equiv 4 \pmod{25}$.

Soln:- Here $a = 11$ $b = 4$ $m = 25$

$\gcd(11, 25) = \gcd = 1 = d$

check $d | b$

$\Rightarrow 1 | 4$ true

$\therefore$ given congruence has unique solution

consider $11x \equiv 4 \pmod{25}$

$\Rightarrow 11x - 4 = 25 \times K$

$11x = 25K + 4$

$x = \dfrac{25K + 4}{11}$

put $K = 0$, $x = \frac{4}{11} \notin Z$

$K = 1$, $x = 29/11 \notin Z$

$K = 2$, $x = 54/11 \notin Z$

$K = 3$, $x = 79/11 \notin Z$

$K = 4$, $x = 104/11 \notin Z$

$K = 5$, $x = 129/11 \notin Z$

$K = 6$, $x = \frac{154}{11} = 14 \in Z$

$\therefore$ $x \equiv 14 \pmod{25}$

Q Encrypt the message STOP using RSA with key (2537, 13) using the prime numbers 43 and 59.

Soln: Given $p = 43$, $q = 59$ and public key $= \{2537, 13\} = \{n, e\}$

$\therefore n = pq = 43 \times 59 = 2537$ and $e = 13$.

$\therefore \phi(n) = (p-1)(q-1) = 42 \times 58 = 2436$

Since $e = 13$, and $1 < e < \phi(n)$ i.e $1 < 13 < 2436$, $\gcd(2436, 13) = 1$.

$M = STOP = \underline{1819}\underline{1415}$ $\begin{bmatrix} 0\,1\,2\,3\,4\,5 \\ A\,B\,C\,D\,E\,F \ldots \end{bmatrix}$

Let $M_1 = 1819$, $M_2 = 1415$

Encryption : $C = M^e \text{ Mod } n$.

$\Rightarrow C_1 = M_1^e \bmod n$ and $C_2 = M_2^e \bmod n$.

$\Rightarrow C_1 = (1819)^{13} \pmod{2537} \rightarrow ①$

consider $(1819)^3 \equiv 2068 \pmod{2537}$ (using calculator) $\rightarrow *$

cube on both sides

$\Rightarrow ((1819)^3)^3 \equiv (2068)^3 \pmod{2537}$

$\Rightarrow 1819^9 \equiv 322 \pmod{2537} \rightarrow ②$ (using calci)

eqn $*$ $\times$ eqn ② $\Rightarrow (1819)^3 (1819)^9 \equiv (2068 \times 322) \bmod 2537$.

$\Rightarrow (1819)^{12} \equiv 1202 \pmod{2537}$ (using calci)

$\times^{ly}$ by 1819 on b.s.

$(1819)^{12}(1819) \equiv (1202 \times 1819) \pmod{2537}$

$\Rightarrow (1819)^{13} \equiv 2081 \pmod{2537}$ (using calci)

$\therefore ① \Rightarrow \boxed{C_1 = 2081 \pmod{2537}}$

Now $C_2 = M_2^e \bmod n$.

$\Rightarrow C_2 = (1415)^{13} \pmod{2537} \rightarrow ③$.

consider $(1415)^3 \equiv 1828 \pmod{2537}$ (using calculator) $\rightarrow **$

$\Rightarrow ((1415)^3)^3 \equiv 1828^3 \pmod{2537}$

$\Rightarrow (1415)^9 \equiv 2005 \pmod{2537} \rightarrow ④$ (using calci)

eqn $**$ $\times$ eqn ④ $\Rightarrow (1415)^3 (1415)^9 \equiv (1828 \times 2005) \pmod{2537}$

$\Rightarrow (1415)^{12} \equiv 1712 \pmod{2537}$ (using calci)

$\times^{ly}$ by 1415 on b.S

$(1415)^{12}(1415) \equiv (1415 \times 1712) \pmod{2537}$

$\Rightarrow (1415)^{13} \equiv 2182 \pmod{2537}$

$\therefore$ eq^n 3 becomes

$$\boxed{C_2 \equiv 2182 \pmod{2537}}$$

Thus $C = C_1 C_2$

$C = \underline{20}\,\underline{8}\,\underline{1}\;\underline{21}\,\underline{8}\,\underline{2}$

$\underline{C = UHBVHC}$

08) a) using Fermat's little theorem, show that $8^{30}-1$ is divisible by 31.

Soln: Here $a = 8$ $p = 31$

By FLT, $a^{p-1} \equiv 1 (\bmod (p))$

$$8^{30} \equiv 1 (\bmod 31)$$

$$8^{30} - 1 \equiv 0 (\bmod 31)$$

$8^{30}-1$ is having a remainder zero.

when it is divided by 31

i.e $8^{30}-1$ is divisible by 31.

b) Solve the system of linear congruence.

$x \equiv 3 (\bmod 5)$, $y \equiv 2 (\bmod 6)$, $z \equiv 4 (\bmod 7)$ using Remainder Theorem.

Soln:- Here $b_1 = 3$ $b_2 = 2$ $b_3 = 4$

$m_1 = 5$ $m_2 = 6$ $m_3 = 7$

we see that $(m_1, m_2) = (5,6) = 1$

$(m_1, m_3) = (5,7) = 1$

$(m_2, m_3) = (6,7) = 1$

Now, $M = m_1 \cdot m_2 \cdot m_3$

$M = 5 \times 6 \times 7 = 210$

and $M_k = \frac{M}{m_k}$; $k = 1, 2, 3$

$M_1 = \frac{210}{5} = 42$, $M_2 = \frac{210}{6} = 35$, $M_3 = \frac{210}{7} = 30$

consider $M_k X \equiv 1 \pmod{m_k}$; $k = 1, 2, 3$, $X = x, y, z$

$42x \equiv 1 \pmod 5$ $\quad$ $35y \equiv 1 \pmod 6$ $\quad$ $30z \equiv 1 \pmod 7$

By inspection,

$x_1 = 3$; $\quad y = 5 \quad z = 4$

thus, $X = M_1 b_1 x + M_2 b_2 y + M_3 b_3 z$ modulo M

$x = 42(3)(3) + 35(2)(5) + 30(4)(4)$ modulo 210

$X = 1208$ (modulo 210)

$X = 158 \pmod{210}$ is the unique solution.

c) i) Find the remainder when $175 \times 113 \times 53$ is divisible by 11.

Soln:- $175 = 10 \pmod{11}$

$113 = 3 \pmod{11}$

$53 = 9 \pmod{11}$

$\therefore 175 \times 113 \times 53 = (10 \times 3 \times 9) \pmod{11}$

$= 270 \pmod{11}$

$= 6 \pmod{11}$

```
11) 270 ( 24
    22
    ---
    050
     44
    ---
     06
    ---
```

$\therefore$ 6 is the remainder.

ii) solve $x^3 + 5x + 1 \equiv 0 \pmod{27}$

let $f(x) = x^3 + 5x + 1 \equiv 0 \pmod{27}$

Soln lies in set $\{0, 1, 2, 3, 4 \ldots\ldots, 27\}$

$f(0) = 1$

$f(1) = 7 \not\equiv 0 \pmod{27}$

$f(2) = 19 \not\equiv 0 \pmod{27}$

$f(3) = 43 \not\equiv 0 \pmod{27}$

$f(4) = 85 \not\equiv 0 \pmod{27}$

$f(5) = 151 \not\equiv 0 \pmod{27}$

$f(6) = 247 \not\equiv 0 \pmod{27}$

$f(7) = 379 \not\equiv 0 \pmod{27}$

$f(8) = 553 \not\equiv 0 \pmod{27}$

$f(9) = 775 \not\equiv 0 \pmod{27}$

$f(10) = 1,051 \not\equiv 0 \pmod{27}$

$f(11) = 1387 \not\equiv 0 \pmod{27}$

$f(12) = 1789 \not\equiv 0 \pmod{27}$

$f(13) = 2263 \not\equiv 0 \pmod{27}$

$f(14) = 2815 \not\equiv 0 \pmod{27}$

$f(15) = 3451 \not\equiv 0 \pmod{27}$

$f(16) = 4177 \not\equiv 0 \pmod{27}$

$f(17) = 4999 \not\equiv 0 \pmod{27}$

$f(18) = 5923 \not\equiv 0 \pmod{27}$

$f(19) = 6955 \not\equiv 0 \pmod{27}$

$f(20) = 8101 \not\equiv 0 \pmod{27}$

$\cancel{f(20)}$

$f(21) = 9367 \not\equiv 0 \pmod{27}$

$f(22) = 10,759 \not\equiv 0 \pmod{27}$

$f(23) = 12283 \not\equiv 0 \pmod{27}$

$f(24) = 13945 \not\equiv 0 \pmod{27}$

$f(25) = 15,751 \not\equiv 0 \pmod{27}$

$f(26) = 17,707 \not\equiv 0 \pmod{27}$

$\therefore x^3 + 5x + 1$ has no solution.

Module - 5 :-

Q.09

a) Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Ans:- Let us consider a matrix A,

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \longleftrightarrow R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$, $R_4 \to 5R_4 - 9R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{5}R_2$, $R_3 \to \frac{1}{33}R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/2 & 7/5 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\therefore \rho(A) = 3}$

b) Solve the system of equations by using Gauss - Jordan method

$x+y+z=9$

$2x+y-z=0$

$2x+5y+7z=52$.

Ans:- Let us consider the Augmented matrix $[A:B]$,

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array}\right]$$

$R_1 \to R_1 + R_2$, $R_3 \to R_3 + 3R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_1 \to 4R_1 - 2R_3$, $R_2 \to 4R_2 - 3R_3$.

$$[A:B] \sim \left[\begin{array}{ccc:c} 4 & 0 & 0 & 4 \\ 0 & -4 & 0 & -12 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$\Rightarrow$ The system of equations are :-

$4x=4$

$\boxed{x=1}$

$-4y=-12$

$y = \frac{\cancel{-}\cancel{12}\,3}{\cancel{-}\cancel{4}}$

$\boxed{y=3}$

$-4z=-20$

$z = \frac{\cancel{-}\cancel{20}\,5}{\cancel{-}\cancel{4}}$

$\boxed{z=5}$

(c) Using power method, find the largest eigen value and corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans: Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -0.333 \\ 0.333 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 0.333 \end{bmatrix} = \begin{bmatrix} 7.332 \\ -3.332 \\ 3.332 \end{bmatrix} = 7.332 \begin{bmatrix} 1 \\ -0.454 \\ 0.454 \end{bmatrix} \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.454 \\ 0.454 \end{bmatrix} = \begin{bmatrix} 7.816 \\ -3.816 \\ 3.816 \end{bmatrix} = 7.816 \begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 7.952 \\ -3.952 \\ 3.952 \end{bmatrix} = 7.952 \begin{bmatrix} 1 \\ -0.497 \\ 0.497 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.497 \\ 0.497 \end{bmatrix} = \begin{bmatrix} 7.988 \\ -3.988 \\ 3.988 \end{bmatrix} = 7.988 \begin{bmatrix} 1 \\ -0.499 \\ 0.499 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.499 \\ 0.499 \end{bmatrix} = \begin{bmatrix} 7.996 \\ -3.996 \\ 3.996 \end{bmatrix} = 7.996 \begin{bmatrix} 1 \\ -0.500 \\ 0.500 \end{bmatrix}$$

Thus after 6 iteration, the approximate Eigen value is $\lambda = 7.996$ and the corresponding eigen vector is $x = \begin{bmatrix} 1 \\ -0.500 \\ 0.500 \end{bmatrix}$

Q.10
a) Solve the following system of eqn by gauss-seidel method

$20x+y-2z=17$ — (1)

$3x+20y-z=-18$ — (2)

$2x-3y+20z=25$ — (3)

Ans:- the given system of eq'ns are diagonally dominant. gauss seidel method is given by

$$x=\frac{17-y+2z}{20} \quad \text{— (4)}$$

$$y=\frac{-18-3x+z}{20} \quad \text{— (5)}$$

$$z=\frac{25-2x+3y}{20} \quad \text{— (6)}$$

let the initial approximation be $(x^{(0)}, y^{(0)}, z^{(0)})=(0,0,0)$

Ist Iteration:-

$$x^{(1)}=\frac{17-0+0}{20}=0.85$$

$$y^{(1)}=\frac{-18-3\times0.85+0}{20}=-1.0275$$

$$z^{(1)}=\frac{25-2\times0.85-3\times1.0275}{20}=1.0109$$

$\therefore (x^{(1)}, y^{(1)}, z^{(1)})=(0.85, -1.0275, 1.0109)$

IInd Iteration:-

$$x^{(2)}=\frac{17+1.0275+2\times1.0109}{20}=1.0025$$

$$y^{(2)}=\frac{-18-3\times1.0025+1.0109}{20}=-0.9998$$

$\therefore (x^{(2)}, y^{(2)}, z^{(2)})=(1.0025, -0.9998, 0.9998)$

III. Iteration:-

$$x(3) = \frac{17 + 0.9998 + 2(0.9998)}{20} = 1.0000$$

$$y(3) = \frac{-18 - 3(1.0000) + 0.9998}{20} = -1.0000$$

$$z(3) = \frac{25 - 2(1.0000) + 3(-1.0000)}{20} = 1.0000$$

$\therefore (x(3), y(3), z(3)) = (1.0000, -1.0000, 1.0000)$

$\therefore$ Thus after three iterations the exact solution for the given system of eqn is $x = 1$, $y = -1$ and $z = 1$

b) Test for consistency.

$x - 2y + 3z = 2.$

$3x - y + 4z = 4$

$2x + y - 2z = 5$ and hence solve.

Ans:- consider the augmented matrix $[A:B]$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 3 & -1 & 4 & : & 4 \\ 2 & 1 & -2 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 5 & -8 & : & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 0 & -3 & : & 3 \end{bmatrix} \quad \text{— (B).}$$

$\therefore \rho(A) = 3$ and $\rho[A:B] = 3$

Thus $\rho(A) = \rho[A:B] = r = 3$

=> system of eqn is consistent

Here number of unknown $= n = 3$

Since $r = n = 3$, system of eqn will have unique soln

From eqn ①
$$x - 2y + 3z = 2.$$
$$5y - 5z = -2$$
$$-3z = 3$$

After solving, $\boxed{z = -1}$

$\Rightarrow 5y - 5z = -2$

$5y - 5(-1) = -2$

$5y + 5 = -2$

$5y = -2 - 5$

$5y = -7$

$\boxed{y = -7/5}$

$\Rightarrow x - 2y + 3z = 2$

$x - 2(-7/5) + 3(-1) = 2.$

$x + 14/5 - 3 = 2.$

$x + 14/5 = 5$

$x = 5 - \frac{14}{5}$

$x = \frac{25-14}{5} = 11/5$

$\boxed{x = 11/5}$

$\therefore$ unique soln is $x = 11/5$, $y = -7/5$ and $z = -1$

Q Solve the system of equations by Gauss elimination method

$2x + y + 4z = 12$

$4x + 11y - z = 33$

$8x - 3y + 2z = 20$

Soln: Consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$R_2 \to R_2 \to 2R_1$, $\quad R_3 \to R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$R_3 \to 9R_3 + 7R_2$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & 0 & -189 & : & -189 \end{bmatrix}$$

this in in upper triangular matrix

=> system of eqn are

$2x + y + 4z = 12$

$9y - 9z = 9$

$-189z = -189$

solving $\boxed{z = 1}$

=> $9y - 9z = 9$

$9y - 9(1) = 9$

$9y = 9 + 9$

$y = 18/9$

$\boxed{y = 2}$

=> $2x + y + 4z = 12$

$2x + 2 + 4(1) = 12$

$2x + 6 = 12$

$2x = 12 - 6$

$2x = 6$

$x = 6/2$

$\boxed{x = 3}$

$\therefore$ Solution is $x = 3$, $y = 2$ and $z = 1$