# Module-5: Linear Algebra

**SYLLABUS:**

**Introduction of linear algebra related to Computer Science &Engineering. Elementary row transformation of a matrix, Rank of a matrix. Consistency and Solution of system of linear equations - Gauss-elimination method, Gauss-Jordan method and approximate solution by Gauss-Seidel method. Eigenvalues and Eigenvectors, Rayleigh's power method to find the dominant Eigenvalue and Eigenvector.**



## Matrix :-

The set of mn number of elements or objects arranged as a 'm' number of rows and 'n' number of columns is called the matrix.

The matrix can be denoted by the capitals of alphabets and the order of the matrix can be denoted as mxn.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

(or)
$A = [a_{ij}]_{m \times n}$ $\quad 1 \le i \le m, \quad i \le j \le n$

## Rank of a Matrix

The number of linearly independent rows or linearly independent columns of a matrix is called the rank of a matrix.

Also, the rank of a matrix in simple words may be explained as the number of non-zero rows (or) columns of a non-zero matrix is called the rank of a matrix.

The highest order of non-zero minor of a matrix is also said to be a rank of the matrix.

The rank of the matrix 'A' can be denoted by $\rho(A)$ and which is $\rho(A) = r$ and it should be always $1 \le \rho(A) \le m$ (or) $1 \le r \le m$

Generally, the rank of a matrix can be evaluated in various ways, they are echelon form.

## Rank of a Matrix Using Echelon form.

1. Let 'A' be a matrix of a order $m \times n$.
2. There are any zero rows then they should be placed below non-zero rows.

3. The number of zero in front of any row increases according to the row number.

4. The non-zero rows of a echelon matrix are that matrix's linearly independent row vectors.

5. Hence, the number of non-zero rows of a matrix reduced in echelon form is called the rank of the matrix and which is $\rho(A)$ or $r(A)$

Key

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

1. Find the rank of a matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 : R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 : R_3 - R_2$

2. Find the rank of the matrix.

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & 5 & 3 & -1 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 + R_1$

$R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 3 & -2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 5 & 2 & -3 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & -2 & -3 \\ 0 & 5 & 2 & -3 \end{bmatrix}$$

$R_3 : R3 - 3R_1$

$R_4 : R_4 - 5R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & -8 & -3 \end{bmatrix} \quad R_4 : R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

3. Find the rank of the matrix

$$\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \\ R_4 : R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$R_4 : R_4 + R_2$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_4 : R_4 - 2R_3$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4$

4. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 3R_1$

$R_4 : R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & -2 \\ 0 & -3 & -7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 : R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \ \rho(A) = 2$

5. Find the rank of the matrix

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 : R_3 - 3R_1$

$R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 : R_3 - R_2$
$R_4 : R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

6. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

=> Let,

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$
$R_3 : R_3 - 4R_1$
$R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} \quad R_4 : R_4 - 4R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

7. Find the rank of the matrix

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - 4R_1 \\ R_4 : R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -19 \end{bmatrix} \quad R_4 : R_4 - 3R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\rho(A) = 4$

8. Find the rank of the matrix

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$$

$\Rightarrow$ let $A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$

$R_2 : R_2 - R_1$
$R_3 : R_3 - R_1$
$R_4 : R_4 - R_1$

$$\sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$R_3 : R_3 - 2R_2$
$R_4 : R_4 - 3R_2$

$$\sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

## Testing of Consistency for System of Linear Equations :-

1. Let 3 system of linear equations with 3 - unknown are :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is called co-efficient matrix.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ the variable (or) unknown matrix

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is called the constant Matrix.

2. Write an argumented matrix

$[A:B] = [A/B] = [AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and reduce it to echelon form.

3. If $\rho(A) = \rho(AB) = r = n$ [no. of unknowns], then the system of equations are consistency and may have unique solutions.

If $\rho(A) = \rho(AB) < n$, then the system of equations are consistency and may have infinite number of solutions.

If $\rho(A) \neq \rho(AB)$ [or] $\rho(A) < \rho(AB)$, then the system of equations are said to be inconsistency and no solutions.

1. Investigate the values of $\lambda$ and $\mu$, so that the equation $2x+3y+5z=9$, $7x+3y-2z=8$ and $2x+3y+\lambda z=\mu$ may have

i. Unique Solution
ii. Many Solution
iii. No Solution

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$R_2 : 2R_2 - 7R_1$
$R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

i) The given system of equation may have unique solution only at,

$$\lambda \neq 5 \;,\; \mu \neq 9$$

ii) The system of equation may have infinite number of solution only at,

$$\lambda = 5 \;,\; \mu = 9$$

iii) The system of equation are inconsistency and with no solution only at,

$$\lambda = 5 \;,\; \mu \neq 9$$

2. For what values of $\lambda$ and $\mu$, the system of equations, $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+\lambda z=\mu$ have
i) Unique Solutions
ii) Infinite
iii) No Solution

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ \mu \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & \lambda & : & \mu \end{bmatrix}$$

$R_2 : R_2 - R_1$
$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & \lambda-6 & \mu-12 \end{array}\right] \quad R_3 : R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & \lambda-8 & \mu-15 \end{array}\right]$$

i. The given solution of equations may have unique solution only at

$$\lambda \neq 8 \quad , \quad \mu \neq 15$$

ii. The given system of equations may have infinite solutions only at,

$$\lambda = 8 \quad , \quad \mu = 15$$

iii The given system of equations are inconsistency only at,

$$\lambda = 8 \; , \; \mu \neq 15$$

Gauss - Elimination Method

1. Test for consistency and solve $x+y+z=6$, $x-y+2z=5$ , $3x+y+z=8$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

$$\therefore [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$

$R_2 : R_2 - R_1$

$R_3 : R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

$\therefore$ The given equations are consistency and follows unique solution.

$$\therefore \; AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \end{bmatrix}$$

$\Rightarrow \quad x + y + z = 6 \text{ ——(1)}$

$\Rightarrow \quad -2y + z = -1 \text{ ——(2)}$

$\Rightarrow \quad -3z = -9 \text{ ——(3)}$

$\therefore z = 3$

$(2) \Rightarrow -2y + 3 = -1$

$\Rightarrow \quad -2y = -4$

$\Rightarrow \quad y = 2$

$(1) \Rightarrow \quad x + y + z = 6$

$\Rightarrow \quad x + 2 + 3 = 6$

$\Rightarrow \quad x = 6 - 5$

$\Rightarrow \quad x = 1$

$\therefore \quad x = 1, \; y = 2, \; z = 3$

2. Solve the system of equation by Guass - elimination method $x+y+z=9$, $2x+y-z=0$, $2x+5y+7z=52$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 52 \end{bmatrix}$$

$$AX = B$$

$$\therefore [A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array}\right]$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array}\right]$$

$R_3 : R_3 + 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$$\rho(A) = \rho(AB) = 3 = n$$

$\therefore$ The system of equation follows unique solution.

$$\therefore AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix}$$

$\Rightarrow \quad x+y+z=9 \quad \text{---} (1)$

$\Rightarrow \quad -y-3z=-18 \quad \text{---} (2)$

$\Rightarrow \quad -4z=-20 \quad \text{---} (3)$

$\therefore \quad z=5$

$(2) \Rightarrow \quad -y-3(5)=-18$

$-y=-3$

$y=3$

$(1) \Rightarrow \quad x+y+z=9$

$x+3+5=9$

$x=9-8$

$x=1$

$\therefore \quad x=1, \; y=3, \; z=5$

3. Test for consistency and solve $5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$AX=B$$

$$\therefore [A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_2 : 5R_2 - 3R_1$

$R_3 : 5R_3 - 7R_1$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 33 \\ 0 & -11 & 1 & : & 3 \end{bmatrix}$$

$R_2 = \frac{1}{11} R_2$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix} \quad R_3 : R_3 + R_2$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 2 = n < 3$$

$\therefore$ The given equation follows infinite solutions

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore \quad 5x + 3y + 7z = 4 \quad \text{---(1)}$$

$$11y - z = 3 \quad \text{---(2)}$$

Let, $z = K$

(2) $\Rightarrow$ $11y - K = 3$

$11y = K + 3$

$y = \frac{1}{11}(K+3)$

(1) $\Rightarrow$ $5x + 3\left[\frac{1}{11}(K+3)\right] + 7K = 4$

$\Rightarrow$ $5x + \frac{3}{11}(K+3) + 7K = 4$

$\Rightarrow$ $55x + 3K + 9 + 7K = 44$

$\Rightarrow$ $55x + 10K = 44 - 9$

$\Rightarrow$ $55x = 35 - 10K$

$\Rightarrow$ $x = \frac{1}{55}(35 - 10K)$

4. Solve the system of equation by Guass - elimination method, $x+2y+z=3$, $3x+2y+z=3$, $x-2y-5z=1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : 3 \\ 3 & 2 & 1 & : 3 \\ 1 & -1 & -5 & : 1 \end{bmatrix}$$

$R_2 : R_2 - 3R_1$

$R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : 3 \\ 0 & -1 & -2 & : -6 \\ 0 & -4 & -6 & : -2 \end{bmatrix}$$

$R_3 : 4R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : 3 \\ 0 & -4 & -2 & : -6 \\ 0 & 0 & 2 & : 2 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

$\therefore$ The given equations follows unique solutions.

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$x + 2y + z = 3 \quad \text{——(1)}$$
$$-4y - 2z = -6 \quad \text{——(2)}$$
$$2z = -2 \quad \text{——(3)}$$

$\therefore \quad z = -1$

(2) $\Rightarrow$ $-4y - 2(-1) = -6$

$-4y + 2 = -6$

$\cancel{-}4y = \cancel{-}8$

$y = 2$

(3) $\Rightarrow$ $x + 2(2) + (-1) = 3$

$x + 4 - 1 = 3$

$x + 3 = 3$

$x = 0$

$\therefore \; x = 0, \; y = 2, \; z = -1$

## Guass - Jordon Method

1. Solve the system of equation by Guass-Jordon method $x+y+z=9$, $x-2y+3z=8$, $2x+y-z=3$.

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

$AX = B$

$$\therefore [A|B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 : R_2 - R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

$R_2 : R_2 - R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 3 & 3 & 3 & 27 \\ 0 & -3 & 2 & -1 \\ 0 & 3 & 9 & 45 \end{array}\right]$$

$R_1 : 3R_1$

$R_3 : -3R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$R_3 : \frac{1}{11} R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 : R_1 - 5R_3$

$R_2 : R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 0 & 6 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 : \frac{1}{3} R_1$

$R_2 : -\frac{1}{3} R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore \; \rho(A) : \rho(AB) = 3 = n$

$AX = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \therefore \; x = 2, \; y = 3, \; z = 4$$

2. Solve the system of equation by Guass - Jordon method $x+y+z=11$, $3x-y+2z=12$, $2x+y-z=3$

$\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 13 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 3 & -1 & 2 & 12 \\ 2 & 1 & -1 & 13 \end{array}\right]$$

$R_2 : R_2 - 3R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 11 \\ 0 & -4 & -1 & -21 \\ 0 & -1 & -3 & -19 \end{array}\right]$$

$R_1 : 4R_1$

$R_3 : -4R_3$

$$\sim \left[\begin{array}{ccc:c} 4 & 4 & 4 & 44 \\ 0 & -4 & -1 & -21 \\ 0 & 4 & 12 & 76 \end{array}\right]$$

$R_1 : R_1 + R_2$

$R_3 : R_3 + R_2$

$$\sim \left[\begin{array}{ccc:c} 4 & 0 & 3 & 23 \\ 0 & -4 & -1 & -21 \\ 0 & 0 & 11 & 55 \end{array}\right]$$

$R_2 : (-1)R_2$

$R_3 : \frac{1}{11} R_3$

$$\sim \left[\begin{array}{ccc:c} 4 & 0 & 3 & 23 \\ 0 & 4 & 1 & 21 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 : R_1 - 3R_3$
$R_2 : R_2 - R_3$

$$\sim \left[\begin{array}{ccc:c} 4 & 0 & 0 & 8 \\ 0 & 4 & 0 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 : \frac{1}{4} R_1$
$R_2 : \frac{1}{4} R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$\therefore\ x = 2,\ y = 4,\ z = 5$

3. Solve the system of equation by Guass-Jordon method $x+y+z=10$, $2x-y+3z=19$, $x+2y+3z=22$

$\Rightarrow$
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \\ 22 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right]$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right]$$

$R_1 : 3R_1$

$R_3 : 3R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 3 & 3 & 30 \\ 0 & -3 & 1 & -1 \\ 0 & 3 & 6 & 36 \end{array}\right]$$

$R_1 : R_1 + R_2$

$R_3 : R_3 + R_2$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array}\right]$$

$R_3 : \frac{1}{7} R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 : R_1 - 4R_3$

$R_2 : R_2 - R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 0 & 9 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 : \frac{1}{3} R_1$

$R_2 : -\frac{1}{3} R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore \; x=3, \; y=2, \; z=5$

4. Solve the system of equation by Guass - Jordon method $x+y+z=8$, $-x-y+2z=-4$, $3x+5y-7z=14$

$\Rightarrow \; \Rightarrow x+y+z=8$ ——(1)

$\Rightarrow -x-y+2z=-4$

$\Rightarrow x+y-2z=4$ ——(2)

$\Rightarrow 3x+5y-7z=14$ ——(3)

$$\Rightarrow [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & 1 & -2 & : & 4 \\ 3 & 5 & -7 & : & 14 \end{bmatrix}$$

$R_2 : R_2 - R_1$

$R_3 : R_3 - 3(R_1)$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 0 & -3 & : & -4 \\ 0 & 2 & -10 & : & -10 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 2 & -10 & -10 \\ 0 & 0 & -3 & -4 \end{array}\right] \quad R_2 : \frac{1}{2} R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & -3 & -4 \end{array}\right] \quad R_1 : R_1 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 6 & 13 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & -3 & -4 \end{array}\right] \quad \begin{array}{l} R_1 : R_1 + 2R_3 \\ R_2 : 3R_2 - 5R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & -3 & -4 \end{array}\right] \quad \begin{array}{l} R_2 : \frac{1}{3} R_2 \\ R_3 : -\frac{1}{3} R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5/3 \\ 4/3 \end{bmatrix}$$

$x = 5, \quad y = 5/3, \quad z = 4/3$

5. Solve the equation by using Guass- Jordon method $x+y+z=9$ , $2x+y-z=0$ , $2x+5y+7z=52$

$\Rightarrow$
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 52 \end{bmatrix}$$

$$AX = B$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array}\right]$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array}\right]$$

$R_3 : R_3 + 3R_1$

$R_1 : R_1 + R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_3 : \left[-\frac{1}{4}\right] R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 : R_1 + 2R_3$

$R_2 : R_2 + 3R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_2 : (-1) R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$x = 1, \; y = 3, \; z = 5$

6. Solve the equation by using Guass- Jordon method $x + 2y + z = 3$, $2x + 3y + 2z = 5$, $3x - 5y + 5z = 2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$AX = B$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 2 & : & 5 \\ 3 & -5 & 5 & : & 2 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$
$R_3 : R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 0 & : & -1 \\ 0 & -11 & 2 & : & -7 \end{bmatrix}$$

$R_1 : R_1 + 2R_2$
$R_3 : R_3 - 11R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 2 & : & 4 \end{bmatrix}$$

$R_1 : 2R_1 - R_3$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right] \quad R_1 : \frac{1}{2} R_1$$

$R_2 : (-1) R_2$

$R_3 : (1/2) R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore x = -1,\ y = 1,\ z = 2$

## Guass - Seidel Method (or) Guass - Seidel Itarative Method

1. Let the 3 system of linear equations are

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \text{ ——— } (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \text{ ——— } (2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \text{ ——— } (3)$$

2. check the property of diagonal dominant as given below.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

3. If the diagonal elements are not in diagonal dominant, then rearrange the equation.

4. Write the unknown $x_1, x_2, x_3$ from the 3 eqn (1) (2) & (3)

$$(1) \Rightarrow \quad x_1 = \frac{1}{a_{11}} \left| b_1 - a_{12}x_2 - a_{13}x_3 \right|$$

$$(2) \Rightarrow \quad x_2 = \frac{1}{a_{22}} \left| b_2 - a_{21}x_1 - a_{23}x_3 \right|$$

$$(3) \Rightarrow \quad x_3 = \frac{1}{a_{33}} \left| b_3 - a_{31}x_1 - a_{32}x_2 \right|$$

5. Take the initial condition as $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ and start iterations by updating the values of $x_1, x_2, x_3$ and continue the same until to reach the solution (approximately)

1. Solve the system of equation by using Guass - Seidel method.

$$10x + y + z = 12, \quad x + 10y + z = 12, \quad x + y + 10z = 12$$

$$\Rightarrow \quad 10x + y + z = 12 \text{ —— (1)}$$

$$x + 10y + z = 12 \text{ —— (2)}$$

$$x + y + 10z = 12 \text{ —— (3)}$$

$\therefore$ The given equations are diagonally dominant

$$(1) \Rightarrow \quad x = \frac{1}{10}[12 - y - z]$$

$$(2) \Rightarrow \quad y = \frac{1}{10}[12 - x - z]$$

$$(3) \Rightarrow \quad z = \frac{1}{10}[12 - x - y]$$

$$I-(1) \Rightarrow x^{(1)} = \frac{1}{10}[12-0-0] = 1.2$$

$$y^{(1)} = \frac{1}{10}[12-1.2-0] = 1.08$$

$$z^{(1)} = \frac{1}{10}[12-1.2-1.08] = 0.972$$

$$I-(2) \Rightarrow x^{(2)} = \frac{1}{10}[12-1.08-0.972] = 0.9948$$

$$y^{(2)} = \frac{1}{10}[12-0.9948-0.972] = 1.0033$$

$$z^{(2)} = \frac{1}{10}[12-0.9948-1.0033] = 1.0002$$

$$I-(3) \Rightarrow x^{(3)} = \frac{1}{10}[12-1.0033-1.0002] = 1$$

$$y^{(3)} = \frac{1}{10}[12-1-1.0002] = 1$$

$$z^{(3)} = \frac{1}{10}[12-1-1] = 1$$

$$I-(4) \Rightarrow x^{(4)} = \frac{1}{10}[12-1-1] = 1$$

$$y^{(4)} = \frac{1}{10}[12-1-1] = 1$$

$$z^{(4)} = \frac{1}{10}[12-1-1] = 1$$

$\therefore$ The solution is

$x=1, \; y=1, \; z=1$

2. Solve the following equation by using Guass- Scidel method.

$$5x+2y+z=12,\ x+4y+2z=15,\ x+2y+5z=20$$

taking (0,0,0) as an initiatial approximation [carry out 4 itaration].

$\Rightarrow$ $5x+2y+z=12$ —— (1)

$x+4y+2z=15$ —— (2)

$x+2y+5z=20$ —— (3)

$\therefore$ The given equations are diagonally dominant

(1) $\Rightarrow$ $x=\frac{1}{5}[12-2y-z]$

(2) $\Rightarrow$ $y=\frac{1}{4}[15-x-2z]$

(3) $\Rightarrow$ $z=\frac{1}{5}[20-x-2y]$

I-(1) $\Rightarrow$ $x^{(1)}=\frac{12}{5}=2.4$

$y^{(1)}=\frac{1}{4}[15-2.4-0]=3.15$

$z^{(1)}=\frac{1}{5}[20-2.4-2(3.15)]=2.26$

I-(2) $\Rightarrow$ $x^{(2)}=\frac{1}{5}[12-2(3.15)-2.26]=0.688$

$y(2)=\frac{1}{4}[15-0.688-2(2.26)]=2.448$

$z(2)=\frac{1}{5}[20-0.688-2(2.448)]=2.8832$

$$I-(3) \Rightarrow \quad x^{(3)} = \frac{1}{5}\left[12 - 2(2.448) - 2.8832\right] = 0.84416$$

$$y^{(3)} = \frac{1}{4}\left[15 - 0.84416 - 2(2.8832)\right] = 2.09736$$

$$z^{(3)} = \frac{1}{5}\left[20 - 0.84416 - 2(2.09736)\right] = 2.9922$$

$$I-(4) \Rightarrow \quad x^{(4)} = \frac{1}{5}\left[12 - 2(2.09736) - 2.9922\right] = 0.9626$$

$$y^{(4)} = \frac{1}{4}\left[15 - 0.9626 - 2(2.9922)\right] = 2.0132$$

$$z^{(4)} = \frac{1}{5}\left[20 - 0.9626 - 2(2.0132)\right] = 3.0022$$

$\therefore$ The solution is

$$x = 0.9626 \backsim 1$$
$$y = 2.0132 \backsim 2$$
$$z = 3.0022 \backsim 3$$

3. $2x - 3y + 20z = 25$, $20x + y - 2z = 17$, $3x + 20y - z = -18$ using Gauss - Seidel taking $(0,0,0)$ as an initial approximation.

$\Rightarrow$ The given equation are not in diagonally dominant form. By reordering the given equation, we have

$$20x + y - 2z = 17 \quad \text{---(1)}$$
$$3x + 20y - z = -18 \quad \text{---(2)}$$
$$2x - 3y + 20z = 25 \quad \text{---(3)}$$

(1) $\Rightarrow$ $x = \frac{1}{20}[17 - y + 2z]$

(2) $\Rightarrow$ $y = \frac{1}{20}[-18 - 3x + z]$

(3) $\Rightarrow$ $z = \frac{1}{20}[25 - 2x + 3y]$

I-(1) $\Rightarrow$ $x^{(1)} = \frac{17}{20} = 0.85$

$y^{(1)} = \frac{1}{20}[-18 - 3(0.85) + 0] = -1.0275$

$z^{(1)} = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0108$

I-(2) $\Rightarrow$ $x^{(2)} = \frac{1}{20}[17 + 1.0275 + 2(1.0108)] = 1.0025$

$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + (1.0108)] = -0.998$

$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.998)] = 0.9998$

I-(3) $\Rightarrow$ $x^{(3)} = \frac{1}{20}[17 + 0.998 + 2(0.9998)] = 1$

$y^{(3)} = \frac{1}{20}[-18 - 3(1) + 0.9998] = -1$

$z^{(3)} = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1$

I-(4) $\Rightarrow$ $x^{(4)} = \frac{1}{20}[17 + 1 + 12] = 1$

$y^{(4)} = \frac{1}{20}[-18 - 3 + 1] = -1$

$z^{(4)} = \frac{1}{20}[25 - 2 - 3] = 1$

$\therefore$ The solution is $x = 1$, $y = -1$, $z = 1$

11. $12x + y + z = 31$ , $2x + 8y - z = 24$ , $3x + 4y + 10z = 58$ using Gauss-Seidel taking $(0,0,0)$ as an initial approximation.

$\Rightarrow$

$$12x + y + z = 31 \quad \text{——— (1)}$$

$$2x + 8y - z = 24 \quad \text{——— (2)}$$

$$3x + 4y + 10z = 58 \quad \text{——— (3)}$$

$$(1) \Rightarrow \quad x = \frac{1}{12}\left[31 - y - z\right]$$

$$(2) \Rightarrow \quad y = \frac{1}{8}\left[24 - 2x + z\right]$$

$$(3) \Rightarrow \quad z = \frac{1}{10}\left[58 - 3x - 4y\right]$$

$$I-(1) \Rightarrow \quad x^{(1)} = \frac{31}{12} = 2.5833$$

$$y^{(1)} = \frac{1}{8}\left[24 - 2(2.5833) + 0\right] = 2.3542$$

$$z^{(1)} = \frac{1}{10}\left[58 - 3(2.5833) - 4(2.3542)\right] = 4.0833$$

$$I-(2) \Rightarrow \quad x^{(2)} = \frac{1}{12}\left[31 - 2.3542 - 4.0833\right] = 2.0468$$

$$y^{(2)} = \frac{1}{8}\left[24 - 2(2.7274) + 4.0833\right] = 1.1923$$

$$z^{(2)} = \frac{1}{10}\left[58 - 3(2.7274) - 4(1.1923)\right] = 4.8626$$

$$I-(3) \Rightarrow \quad x^{(3)} = \frac{1}{12}\left[31 - 1.1923 - 4.8686\right] = 2.0787$$

$$y^{(3)} = \frac{1}{8}\left[24 - 2(2.8892) + 4.8626\right] = 2.8855$$

$$z^{(3)} = \frac{1}{10}\left[58 - 3(2.8892) - 4(2.8855)\right] = 3.7190$$

$\therefore$ The solution is $x = 2$ , $y = 3$ , $z = 4$

# Eigen Values And Eigen Vectors

Eigen values are a special set of scalar associated with the linear system of equations they are also known as characteristics roots and characteristics values and they can be determined by taking

$$(A - \lambda I)\, X = 0$$

where, $A$ = Square matrix

$I$ = Unit matrix

$X$ = Variable matrix having single column

And the determinant of $A - \lambda I$ can be used to find the eigen values and which follows as

$|A - \lambda I| = 0$ called characteristic equation and it can provide the roots of '$\lambda$'.

## Rayleigh's Power Method (or) Power Method

To determine the largest eigen value and the respective eigen vector we used to follow the given - working rule.

1. Let the given square matrix as 'A' with the initial eigen vector as 'X' follows:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

2. Take the product of 'A' and the initial eigen vector and bring out the numerically largest value, say it as 'λ'

$$\therefore \quad AX^{(0)} = X^{(1)}\lambda^{(1)}$$

3. Continue the same process with the resultant eigen vectors until to reach an equal eigen value

$$\therefore \quad AX^{(1)} = X^{(2)}\lambda^{(2)}$$

$$AX^{(2)} = X^{(3)}\lambda^{(3)}$$

$$AX^{(3)} = X^{(4)}\lambda^{(4)}$$

$$\vdots \qquad \vdots$$

4. Finally, the obtained 'λ' is called the largest eigen value and the vector 'X' is called respective eigen vector.

1. Find the largest Eigen Value of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ with the initial vector $[1\ 0\ 0]^T$

⇒ Let,

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore \quad AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix}$$

$$= 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore \quad AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore \quad AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9286 \end{bmatrix} = \begin{bmatrix} 2.9286 \\ 0 \\ 2.8571 \end{bmatrix}$$

$$= 2.9286 \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9756 \end{bmatrix} = \begin{bmatrix} 2.9756 \\ 0 \\ 2.9512 \end{bmatrix}$$

$$= 2.9756 \begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$\therefore$ The largest eigen value is $2.9756 \sim 3$ and eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. Using Rayleigh's Power method find the dominant eigen value and the corresponding eigen vector of $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking $[1 \ 0 \ 0]^T$ as an initial eigen vector.

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore \quad AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore \quad AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix}$$

$$= 5.6 \begin{bmatrix} 1 \\ 0.9286 \\ -0.9286 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore \quad AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9286 \\ -0.9286 \end{bmatrix} = \begin{bmatrix} 5.8571 \\ 5.7144 \\ -5.7144 \end{bmatrix}$$

$$= 5.8571 \begin{bmatrix} 1 \\ 0.9756 \\ -0.9756 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore \quad AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9756 \\ -0.9756 \end{bmatrix} = \begin{bmatrix} 5.9513 \\ 5.9024 \\ -5.9024 \end{bmatrix}$$

$$= 5.9513 \begin{bmatrix} 1 \\ 0.9918 \\ -0.9918 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$\therefore\ AX^{(5)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9918 \\ -0.9918 \end{bmatrix} = \begin{bmatrix} 5.9836 \\ 5.9672 \\ -5.9672 \end{bmatrix}$$

$$= 5.9836 \begin{bmatrix} 1 \\ 0.9973 \\ -0.9973 \end{bmatrix}$$

$\therefore$ The largest eigen value is $5.9836 \sim 6$ and eigen vector is $\begin{bmatrix} 1 \\ 0.9973 \\ -0.9973 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ with the initial approximate eigen vector $[1\ 0\ 0]^T$ and carry out 4 iterations.

$\Rightarrow$ Let, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore\ AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix}$$

$$= 2.5 \begin{bmatrix} 1 \\ 0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.4286 \\ 1.8572 \end{bmatrix}$$

$$= 3.4286 \begin{bmatrix} 0.8750 \\ -1 \\ 0.5417 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$\therefore$ The largest eigen value is 3.4286 and eigen vector is $\begin{bmatrix} 0.8750 \\ -1 \\ 0.5417 \end{bmatrix}$

4. Using Rayleigh's Power method find the dominant eighen value and the corresponding eighen vector of $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by taking $[1\ 0\ 0]^T$ as the initial eighen vector

$\Rightarrow$ Let,

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad AX^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 2 \end{bmatrix}$$

$$= 25 \begin{bmatrix} 0.04 \\ 1 \\ 0.08 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore \quad AX^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0.04 \\ 1 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.120 \\ 1.68 \end{bmatrix}$$

$$= 25.2 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore \quad AX^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1777 \\ 1.1332 \\ 1.7332 \end{bmatrix}$$

$$= 25.1777 \begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore \quad AX^{(3)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix}$$

$$= 25.1826 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0684 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore \quad AX^{(4)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0451 \\ 0.0684 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7264 \end{bmatrix}$$

$$= 25.1821 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$\therefore \quad AX^{(5)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix}$$

$$= 25.1821 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$\therefore$ The largest eigen value is $25.1821$ and eigen vector is $\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$

5. Using Power method find the largest eigen value and corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \text{ by taking } [1 \ 1 \ 1]^T$$

$$\Rightarrow \quad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$$

$$= 6 \begin{bmatrix} 1 \\ 0 \\ 0.6667 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$\therefore \quad AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.6667 \end{bmatrix} = \begin{bmatrix} 7.3334 \\ -2.6667 \\ 4.0001 \end{bmatrix}$$

$$= 7.3334 \begin{bmatrix} 1 \\ -0.3636 \\ 0.5455 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$\therefore \quad AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3636 \\ 0.5455 \end{bmatrix} = \begin{bmatrix} 7.8182 \\ -3.6363 \\ 4.0001 \end{bmatrix}$$

$$= 7.8182 \begin{bmatrix} 1 \\ -0.4651 \\ 0.5116 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$\therefore \quad AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4651 \\ 0.5116 \end{bmatrix} = \begin{bmatrix} 7.9534 \\ -0.4651 \\ 0.5116 \end{bmatrix}$$

$$= 7.9534 \begin{bmatrix} 1 \\ -0.4912 \\ 0.5029 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$\therefore \quad AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4912 \\ 0.5024 \end{bmatrix} = \begin{bmatrix} 7.9882 \\ -3.9765 \\ 3.4999 \end{bmatrix}$$

$$= 7.9882 \begin{bmatrix} 1 \\ -0.4978 \\ 0.5007 \end{bmatrix}$$

$\therefore$ The largest eigen value is $7.9882 \sim 8$ and eigen vector is $\begin{bmatrix} 1 \\ -0.4978 \\ 0.5007 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$

## Additional Problems :-

1. For What values of $\lambda$ and $\mu$, the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ has

i. No Solution

ii. Unique Solution

iii. Infinite Solution

$$\Rightarrow \quad x+y+z=6 \quad \text{--- (1)}$$
$$x+2y+3z=10 \quad \text{--- (2)}$$
$$x+2y+\lambda z=\mu \quad \text{--- (3)}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B$$

where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix} \quad \begin{matrix} R_2 : R_2 - R_1 \\ R_3 : R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix} \quad R_3 : R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

i. The system of equation may have unique solution only at $\lambda \neq 3$ and any value of $\mu$.

ii. The system of equation may have infinite number of solution at $\lambda = 3$ and $\mu = 10$

iii. The System of equation may have no solution only at $\lambda = 3$ and $\mu \neq 10$.

2. Solve the following system of equations by using Guass- elimination methods $3x+y+2z=3$, $2x-3y-z=-3$ and $x+2y+z=4$

$\Rightarrow$
$$3x+y+2z=3 \quad \text{---(1)}$$
$$2x-3y-z=-3 \quad \text{---(2)}$$
$$x+2y+z=4 \quad \text{---(3)}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$AX = B$

$$[A:B] \Rightarrow \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

$R_1 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

$R_2 : (-5)R_2$

$R_3 : (-7)R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 35 & 15 & 55 \\ 0 & 35 & 7 & 63 \end{array}\right]$$

$R_3 : R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 35 & 15 & 55 \\ 0 & 0 & -8 & 8 \end{array}\right]$$

$\therefore \quad \rho(A) = \rho(AB) = 3 = n$

$\therefore$ The equations may have unique solutions

$AX = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 35 & 15 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 55 \\ 8 \end{bmatrix}$$

$x + 2y + z = 4$ ——— (4)

$35y + 15z = 55 (\div 5)$

$7y + 3z = 11$ ——— (5)

$-8z = 8$ ——— (6)

$\boxed{z = -1}$

$(5) \Rightarrow \quad 7y + 3(1) = 11$

$7y = 11 + 3$

$7y = 14$

$\boxed{y = 2}$

$(6) \Rightarrow \quad x + 2(2) + (-1) = 4$

$x + 4 - 1 = 4$

$\boxed{x = 1}$

$\therefore$ The solution is $x = 1, y = 2, z = -1$

3. Solve the system of equations by Guass-Jordon method.

$x + y + z = 9 \quad , \quad x - 2y + 3z = 8 \quad , \quad 2x + y - z = 3$

$\Rightarrow \quad x + y + z = 9$ ——— (1)

$x - 2y + 3z = 8$ ——— (2)

$2x + y - z = 3$ ——— (3)

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

$AX = B$

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 : - R_2 - R_1$

$R_3 : R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

$R_1 : 3R_1 + R_2$

$R_3 : 3R_3 - R_2 \Rightarrow R_3 : -\frac{1}{11} R_3$

$$\sim \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

$R_1 : -R_1 - 5R_3$

$R_2 : R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 : \frac{1}{3} R_1$

$R_2 : -\frac{1}{3} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\rho(A) = \rho(AB) = 3 = n$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\boxed{x = 2}$, $\boxed{y = 3}$, $\boxed{z = 4}$

4. Solve the system of equations by Guass - Elimination method.

$x + 2y + z = 3$ , $3x + 2y + z = 3$ , $x - 2y - 5z = 1$

$\Rightarrow$ $x + 2y + z = 3$ —— (1)

$3x + 2y + z = 3$ —— (2)

$x - 2y - 5z = 1$ —— (3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \Rightarrow AX = B$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 3 & 2 & 1 & : & 3 \\ 1 & -2 & -5 & : & 1 \end{bmatrix}$$

$R_2 : R_2 - 3R_1$

$R_3 : R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -4 & -2 & : & -6 \\ 0 & -4 & -6 & : & -2 \end{bmatrix}$$

$R_2 : \left(-\frac{1}{2}\right) R_2$

$R_3 : \left(-\frac{1}{2}\right) R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & 2 & 3 & : & 1 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & 0 & 2 & : & -2 \end{bmatrix}$$

$$\rho(A) = \rho(AB) = 3 = n$$

$\therefore$ The given equations are unique solutions;

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

$$AX = B$$

$x + 2y + z = 3$ —— (1)

$2y + z = 3$ —— (2)

$2z = -2$ —— (3)

$$\boxed{z = -1}$$

(2) $\Rightarrow$ $2y+(-1)=3$

$2y-1=3$

$2y=4$

$\boxed{y=2}$

(3) $\Rightarrow$ $x+2y+z=3$

$x+2(2)-1=3$

$x+4-1=3$

$x=3-3$

$\boxed{x=0}$

5. $83x+11y-4z=95$, $3x+8y+29z=71$, $7x+52y+13z=104$ using Guass-Seidel taking $(0,0,0)$ as initial approximation.

$\Rightarrow$ $83x+11y-4z=95$

$3x+8y+29z=71$

$7x+52y+13z=104$

The given equations are not in diagonally dominant form. By reordering the given equations we have,

$83x+11y-4z=95$ ——(1)

$7x+52y+13z=104$ ——(2)

$3x+8y+29z=71$ ——(3)

(1) $\Rightarrow$ $x=\frac{1}{83}[95-11y+4z]$

$\Rightarrow$ $y=\frac{1}{52}[104-7x-13z]$

$\Rightarrow$ $z=\frac{1}{29}[71-3x-8y]$

I-(1) $x^{(1)}=\frac{95}{83}=1.1446$

$y^{(1)}=\frac{1}{52}[104-7(1.1446)-13(0)]=1.8459$

$z^{(1)}=\frac{1}{29}[71-3(1.1446)-8(1.8459)]=1.8206$

I = 2

$$x^{(2)} = \frac{1}{83}\left[95 - 11(1.8459) + 4(1.8006)\right] = 0.9876$$

$$y^{(2)} = \frac{1}{52}\left[104 - 7(0.9876) - 13(1.8006)\right] = 1.4119$$

$$z^{(2)} = \frac{1}{29}\left[71 - 3(0.9876) - 8(1.4119)\right] = 1.9566$$

I = 3

$$x^{(3)} = \frac{1}{83}\left[95 - 11(1.4119) + 4[1.9566]\right] = 1.0515$$

$$y^{(3)} = \frac{1}{52}\left[104 - 7(1.0515) - 13(1.9566)\right] = 1.3693$$

$$z^{(3)} = \frac{1}{29}\left[71 - 3(1.0516) - 8(1.3693)\right] = 1.9617$$

I = 4

$$x^{(4)} = \frac{1}{83}\left[95 - 11(1.3693) + 4(1.9617)\right] = 1.0576$$

$$y^{(4)} = \frac{1}{52}\left[104 - 7(1.0576) - 13(1.9617)\right] = 1.3672$$

$$z^{(4)} = \frac{1}{29}\left[71 - 3(1.0576) - 8(1.3672)\right] = 1.9617$$

The solution is $\therefore$ $x = 1.0576$

$y = 1.3672$

$z = 1.9617$

6. $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$, Using Guass - Seidel taking (0,0,0) as initial approximation.

$\Rightarrow$
$$27x + 6y - z = 85 \quad \text{—— (1)}$$
$$6x + 15y + 2z = 72 \quad \text{—— (2)}$$
$$x + y + 54z = 110 \quad \text{—— (3)}$$

$$(1) \Rightarrow \quad x = \frac{1}{27}[85 - 6y + z]$$

$$(2) \Rightarrow \quad y = \frac{1}{15}[72 - 6x - 2z]$$

$$(3) \Rightarrow \quad z = \frac{1}{54}[110 - x - y]$$

I-(1)

$$x^{(1)} = \frac{1}{27}[85 - 6(0) + 0] = 3.1481$$

$$y^{(1)} = \frac{1}{15}[72 - 6(3.1481) - 2(0)] = 3.5407$$

$$z^{(1)} = \frac{1}{54}[110 - 3.1481 - 3.5407] = 1.9131$$

I-(2)

$$x^{(2)} = \frac{1}{27}[85 - 6(3.5407) + 1.9131] = 2.4321$$

$$y^{(2)} = \frac{1}{15}[72 - 6(2.4321) - 2(1.9131)] = 3.5720$$

$$z^{(2)} = \frac{1}{54}[110 - 2.4321 - 3.5720] = 1.92585$$

I-(3)

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.5720) + 1.9258\right] = 2.4256$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6(2.4256) - 2(1.9258)\right] = 3.5729$$

$$z^{(3)} = \frac{1}{54}\left[110 - 2.4256 - 3.5729\right] = 1.9259$$

I-(4)

$$x^{(4)} = \frac{1}{27}\left[85 - 6(3.5729) + 1.9259\right] = 2.4255$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6(2.4255) - 2(1.9258)\right] = 3.5730$$

$$z^{(4)} = \frac{1}{54}\left[110 - 2.4255 - 3.5730\right] = 1.9259$$

$\therefore$ The solution are $x = 2.4255$

$y = 3.5730$

$z = 1.9259$

THANK YOU

ALL THE BEST