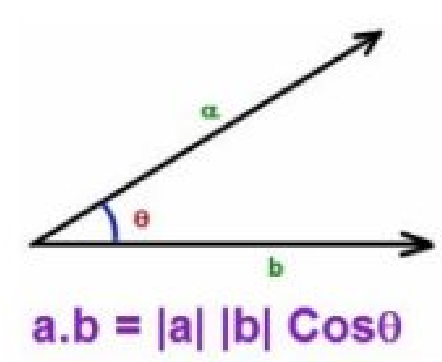
MODULE 4

MAXWELL'S EQUATIONS AND ELECTROMAGNETIC WAVES

Fundamentals of vector calculus

1. Scalar Product or Dot Product

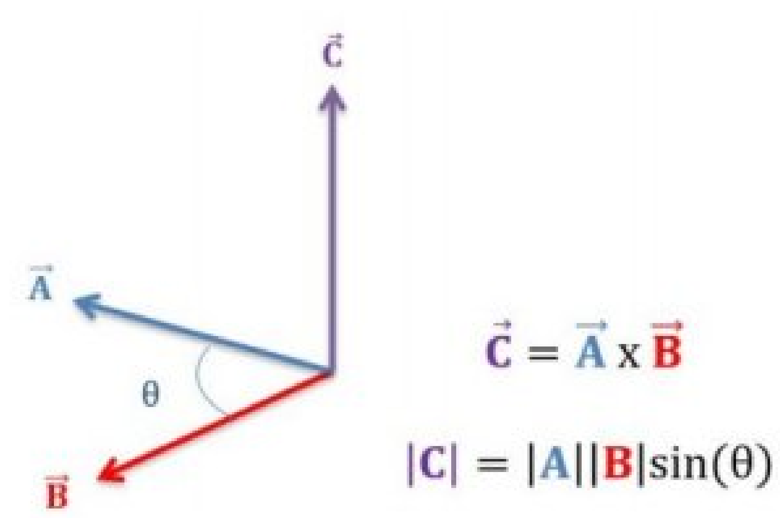
The scalar product of two vectors is an algebraic operation equal to the product of their magnitudes and the cosine of the angle between them.



Example: Mechanical work is a dot product of force and displacement vectors, Power is the dot product of force and velocity, magnetic flux is the dot product of magnetic field and surface area.

2. Vector Product or Cross Product:

The vector product of two vectors is an algebraic operation equal to the product of their magnitudes and the sine of the angle between them.



Example: Torque is the vector product of the force applied to a lever multiplied by its distance from the lever's rigid support.

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Del Operator - ∇

Del is also called as *nabla* which is a vector differential operator used to find the derivative of a vector. If *i, j, k* denote the basis vectors in *x, y, z* directions respectively then the *del* operator is expressed as,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Gradient - grad P

Gradient is the vector derivatives of a scalar field. When a scalar field such as *P(x, y, z)* is multiplied by the *del* operator, the resultant field is a vector field called the gradient and the components of vector at each point are just the partial derivatives of the scalar field at that point. *i.e,*

$$\nabla P(x, y, z) = i\frac{\partial P}{\partial x} + j\frac{\partial P}{\partial y} + k\frac{\partial P}{\partial z}$$

For example: If we think of 'f' as describing the temperature at a point (x, y), then the gradient gives the direction in which the temperature is increasing most rapidly.

Curl

Curl is a vector operator that describes the infinitesimal circulation or rotation of a vector field in three dimensional space. The curl of the point in a field is represented as a vector. The attributes of this vector characterizes the rotation at that point.

Physical Significance of a Curl:

The curl of a vector field at a given point is a measure of how much the field curls (rotates or circulates) around the point. If there is no magnetic field present and consider only static electric field then the curl of electric field $\vec{E}$ will be zero. Hence $\nabla \times \vec{E} = 0$

Divergence

Divergence is the scalar derivative of a vector field. The divergence operation turns a vector field into a scalar field thereby giving the scalar value of the source (a point or region of positive divergence) or sink(a point or region of negative divergence).

Physical Significance of Divergence:

Divergence is roughly a measure of vector field's increase in the direction it points. But more accurately it is a measure of that field's tendency to converge or repel from a given point.

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Linear , surface and Volume Integral

1. Linear Integral: The integral of a vector along a curve is called as line integral.

$$i.e, \quad \int_{P_1}^{P_2} \vec{A}.\vec{dl} = \int_{P_1}^{P_2} A.dl\ cos\theta$$

2. Surface Integral: A surface integral is a generalization of double integrals in integrating over a surface that lies in a dimensional space. The dot product $\vec{A}.d\vec{S}$ represents the flux of the vector field $\vec{A}$ across the surface dS. Hence the total flux of the vector field $\vec{A}$ across the entire surface S is given as

$$surface\ integral = \int\int_s \vec{A}.d\vec{S} = \int\int_s A\ n\ \ ds$$

where 'n' is the unit vector.

3. Volume Integral: A volume integral is an integral where the function is integrated along a volume in a 3-dimensional space. It is a specific type of triple integral.
Consider an elementary volume dV at some point in the solid region, let ρ_v be the charge density at that point. Then the volume integral of ρ_v over the volume is given as

$$\oint_V \rho_v.\ dV = \int\int_v\int \rho_v\ \ dV$$

Gauss Divergence Theorem

Gauss Divergence Theorem states that, the volume integral of the divergence of a vector field $\vec{A}$ over any volume V is equal to the surface integral of $\vec{A}$ taken over the closed surface enclosing volume V.

Derivation: Let the closed surface enclose certain volume V, let us subdivide the volume V into a large number of sub- sections called cells. If the i^{th}

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cell has the volume $\triangle V_i$ and is bounded by the surface S, then we can write,

$$\oint_s \vec{A}.d\vec{S} = \sum_i \oint_s \vec{A}.d\vec{S} = \sum_i \frac{\oint_s \vec{A}.d\vec{S}}{\triangle V_i} \triangle V_i \tag{1}$$

where A is the vector field over a closed surface S.
since the cells are adjacent to each other, the outward flux to one cell is inward to its neighboring cells, thus on every interior surface between the cells, there is cancellation of surface integrals and hence the sum of surface integrals over i^{th} surface is equal to the total surface integral over the entire surface S. *i.e,*

$$\oint_s \vec{A}.d\vec{S} = \frac{\oint_s \vec{A}.d\vec{S}}{\triangle V} \triangle V \tag{2}$$

As the volume shrinks about a point, then the right hand side of the equation (2) gives divergence of vector field $\vec{A}$.
$\therefore$ Taking Lim $\triangle V$ tends to zero in equation(2), then we have

$$Lim_{\triangle V \to 0} = \frac{\oint \vec{A}.d\vec{S}}{\triangle V} = div\ \vec{A} = \nabla.\vec{A}$$

$$\therefore \oint \vec{A}.d\vec{S} = [\nabla.\vec{A}] \triangle V \tag{3}$$

Now considering the entire volume V enclosed by the surface S, equation (3)

$$\Longrightarrow \boxed{\oint \vec{A}.d\vec{S} = \int_v [\nabla.\vec{A}] \triangle V}$$

Stokes Theorem

It states that, the surface integral of curl of a vector field $\vec{A}$ over an open surface S enclosed by closed path L is equal to the line integral of the vector field $\vec{A}$ around the closed path L *i.e,*

$$\int_s \left[\nabla \times \vec{A}\right] .d\vec{S} = \oint_L \vec{A}.d\vec{L}$$

where dL is the perimeter of the total surface S

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Gauss's Flux Theorem-Electrostatics and Magnetostatics

According to Electrostatics:
Gauss's law states that the flux passing through any closed surface is equal to $\frac{1}{\epsilon_0}$ times the total charge 'q' enclosed by that surface. *i.e,*

$$\phi = \frac{1}{\epsilon_o}q$$

According to Magnetostatics:
Gauss's law for magnetism states that, the total magnetic flux through any closed surface is equal to zero.
i.e.,

$$\oint_s \vec{B}.d\vec{S} = 0$$

Ampere's Circuital Law and Maxwell's-Ampere's Law-(Mention No Derivation)

Ampere's law states that the line integral of the magnetic field $\vec{H}$ around any closed path is equal to the direct current enclosed by that path. *i.e,*

$$\oint \vec{H}.d\vec{L} = I \tag{1}$$

The statement is valid if the electric field does not change with time, however the changing electric field generates a magnetic field ,this discrepancy in the Ampere's circuital law was corrected and modified by Maxwell.
The resulting equation is called as Maxwell- Ampere's circuital law.*i.e,*

$$I = \oint_s \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right].d\vec{S}$$

where 'D' is the electric flux density.
$\frac{\partial \vec{D}}{\partial t}$ is the displacement current density.

$$\therefore\ equation\ \ (1) \Longrightarrow \oint \vec{H}.d\vec{L} = \oint_s \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right].d\vec{S} \tag{2}$$

$$But\ \oint \vec{H}.d\vec{L} = \int_s (\nabla \times \vec{H}).d\vec{S} \tag{3}$$

$$from\ \ Stoke's\ \ theorem$$

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$$comparing \;\; (2)(3) \;\; we \; have$$

$$\int_s (\nabla \times \vec{H}).d\vec{S} = \oint_s \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] .d\vec{S}$$

$$or \; \boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

Biot-Savart's Law

States that, at any point 'P' the magnitude of the magnetic field dB through a small current carrying element of length dL is directly proportional to the product of current I, length of the element dL , sine of the angle between the element and the line joining the point to that element and inversely proportional to the square of the distance between the point and the element.

$$d\vec{B} = \frac{\mu_0}{4\pi} . \frac{I \; dL \sin\theta}{r^2}$$

$$or \; in \; vector \; form \quad d\vec{B} = \frac{\mu_0}{4\pi} . \frac{I(d\vec{L} \times \vec{r})}{r^2}$$

where μ_0 is the permeability of the free space, a constant.

Faraday's Law of Electromagnetic Induction-Mention

States that, the magnitude of emf induced in the coil of wire equals the number of turns in the coil times the rate of change of magnetic flux linked with the coil and its direction opposes the flux change.

$$\oint \vec{E}.d\vec{L} = -\frac{d}{dt} \left[\oint_s \vec{B}.d\vec{S} \right]$$

$$\implies \oint \vec{E}.d\vec{L} = \int_s (\nabla \times \vec{E}).d\vec{S} = -\frac{d}{dt} \left[\oint_s \vec{B}.d\vec{S} \right]$$

if the magnetic field is not varying with time, we get

$$\int_s (\nabla \times \vec{E}).d\vec{S} = 0$$

Since d$\vec{S}$ cannot be zero

$$\nabla . \vec{E} = 0$$

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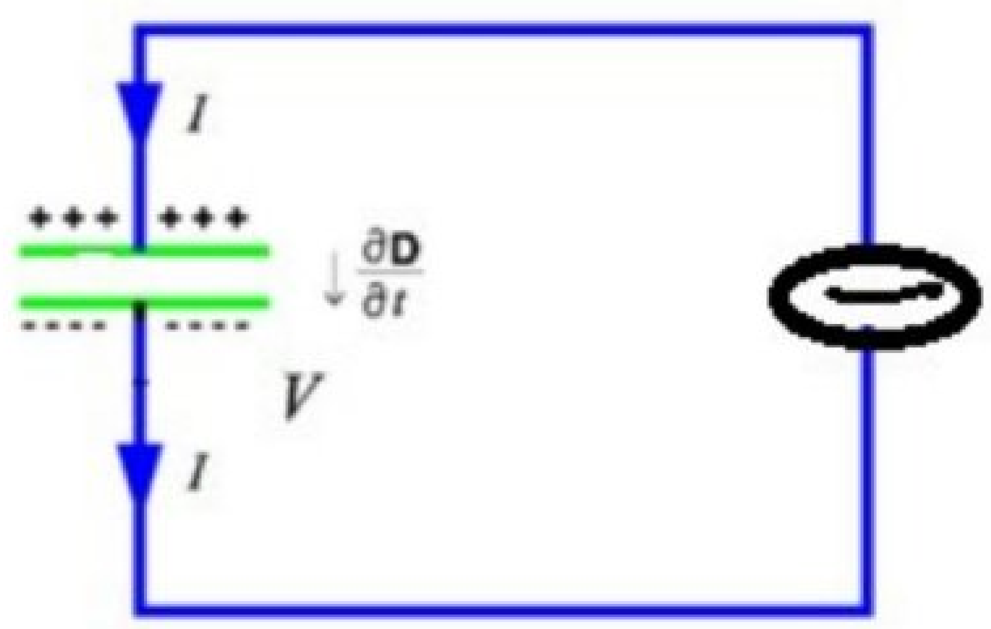
Continuity Equation for Current Density

Continuity equation for electric charges states that the amount of electric charge in any volume of space can only change by the amount of electric current flowing into or out of that volume through its boundaries. Continuity equation is known as the law of conservation of charge.

Displacement Current

Displacement current is the correction factor in Maxwell's equation that appears in time varying condition , but does not describe any movement of charges although it has an associated magnetic field .

Derivation



Consider a parallel plate capacitor connected across an A.C source. Let the area of each plate be *'A'* and *'d'* be their separation distance. Then the total potential for a parallel plate capacitor is given by,

$$V = \frac{E}{d} \;\; or \;\; E = \frac{V}{d} \tag{1}$$

The electric field intensity E across capacitor plates ,

$$E = \frac{D}{\epsilon} \tag{2}$$

$$equating\ equation(1)\ \ and\ \ (2) we\ \ get$$

$$\Longrightarrow \frac{V}{d} = \frac{D}{\epsilon} \;\; or \;\; D = \frac{\epsilon}{d} V_o e^{i\omega t} \tag{3}$$

$$(\text{since } V = V_o\ e^{i\omega t})$$

In general the displacement current density $=\frac{\partial D}{\partial t}$

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By substituting for 'D' and differentiating we get,

$$\therefore \frac{\partial D}{\partial t} = \frac{\epsilon}{d} V_o e^{i\omega t}(i\omega) \tag{4}$$

If I_d is the displacement current then, $\frac{\partial D}{\partial t} = \frac{I_d}{A}$

$$Equation(4) \Longrightarrow \frac{I_d}{A} = \frac{i\omega\epsilon}{d} V_o e^{i\omega t}$$

$$\boxed{\therefore \; displacement \; current, \; I_d = \frac{i\omega A\epsilon}{d} V_o e^{i\omega t}} \tag{5}$$

Maxwell's Equations

1. Using Gauss's law for electrostatic field Maxwell derived an equation, which is called as first Maxwell's equation.

$$\boxed{\nabla.\vec{D} = \rho_v}$$

where D is the electric flux density.
ρ_v is the volume density.

2. According to Gauss's law for magneto statics Maxwell derived an equation, which is called as Second Maxwell's equation.

$$\boxed{\nabla.\vec{B} = 0}$$

where B is the magnetic flux density.

3. According to Maxwell's-Ampere's circuital law, the third Maxwell's equation can be written as,

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

4. From Faraday's law of electromagnetism, Maxwell arrived at fourth Maxwell's equation which is written as,

$$\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}}$$

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Electro-Magnetic Waves

Expression for Electromagnetic Wave Equation in Differential Form Using Maxwell's Equations

Electromagnetic wave equation shows that all waves travel at a speed equal to speed of light.

Consider the Maxwell's Ampere's equation(Maxwell's third law),

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1}$$

$$But \;\; D = \epsilon E \tag{2}$$

$$\therefore \; equation(3) \; becomes \; \nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \tag{3}$$

From Maxwell's fourth law we have,

$$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t} \tag{4}$$

Then the magnetic flux density 'B' is given by,

$$\vec{B} = \mu \; \vec{H}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{5}$$

to eliminate $\vec{H}$ taking curl both sides, we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$from \; equation(3) \; we \; have \; \nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\implies \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$since \; J = 0 \; in \; free \; space$$

$$\nabla \times \nabla \times \vec{E} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{6}$$

$$As \; per \; vector \; identity \; \nabla \times \nabla \times \vec{E} = \nabla(\nabla.\vec{E}) - \nabla^2 \vec{E} \tag{7}$$

$$From \; Gauss's \; law \; \nabla.\vec{D} = \rho_v \;\; (or) \;\; \nabla.\vec{E} = \frac{\rho_v}{\epsilon} \;\; using \; equation(4)$$

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$$\therefore\ equation(7)\ becomes$$

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2\vec{E}$$

$$then\ equation\ (6) \Longrightarrow \nabla\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2\vec{E} = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$since\ \rho_v = 0\ for\ free\ space$$

$$\boxed{\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0} \tag{8}$$

Equation(8) is called as general wave equation in terms of electric field 'E'. Similarly the wave equation in terms of magnetic field 'H' can be written as,

$$\boxed{\nabla^2\vec{H} - \mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0}$$

Expression for electromagnetic Wave Equation in Vacuum in terms of $\mu_o,\ \epsilon_o\ and\ C$ (Mention No derivation)

The equation for plane electromagnetic waves in vaccum in terms of electric field vector and magnetic field vector are written as follows:

$$\nabla^2\vec{E} = \mu_o\epsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

$$where\ \ \mu_o\ \epsilon_o = \frac{1}{c^2}$$

$$\Longrightarrow \boxed{\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$similarly\ in\ terms\ of\ magnetic\ field\ vector$$

$$the\ same\ equation\ can\ be\ written\ as$$

$$\boxed{\nabla^2\vec{H} = \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}}$$

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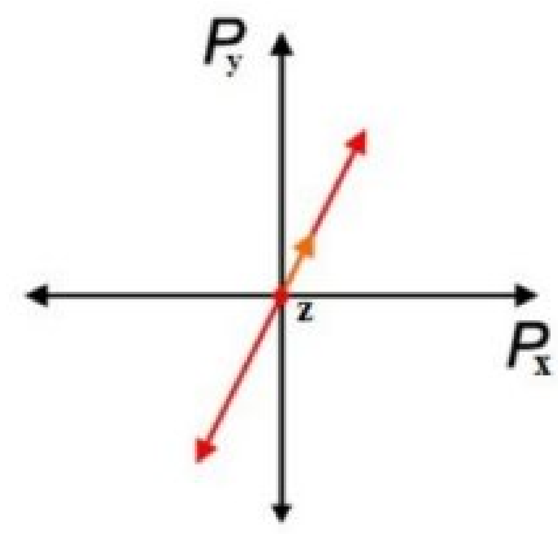
Polarization of Electromagnetic Waves

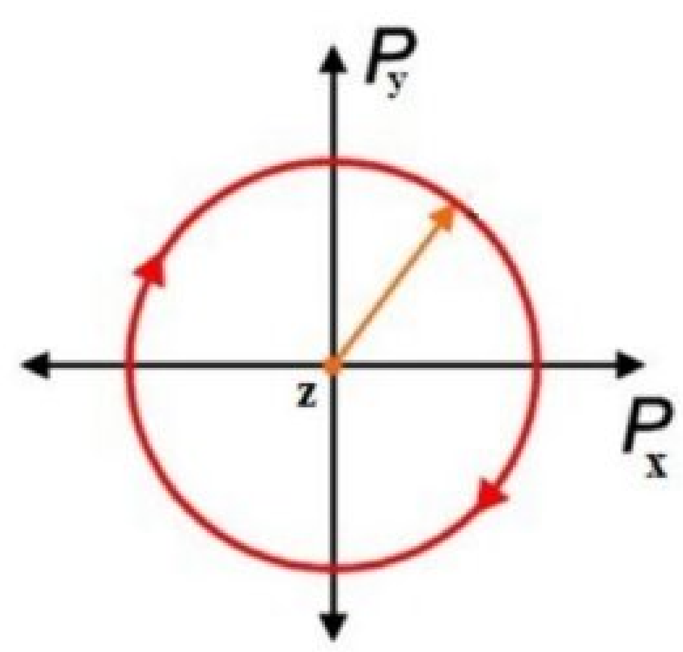
1. Linear Polarization



a

Linear polarization or plane polarization of electromagnetic waves is a confinement of the electric field vector or magnetic field vector to a given plane along the direction of propagation.
The condition for linear polarization for a wave propagating in z-direction is that, the components P_x and P_y must be in phase , with their amplitudes being equal or unequal.

2. Circular Polarization



c

If the electric field having two orthogonal polarized components P_x and P_y of equal amplitudes but different in phase by 90°, then the wave is said to be circularly polarized.
The condition for circular polarization for a wave propagating in Z-direction is that, the electric field components P_x and P_y must have a constant phase difference of 90° and with equal amplitudes.

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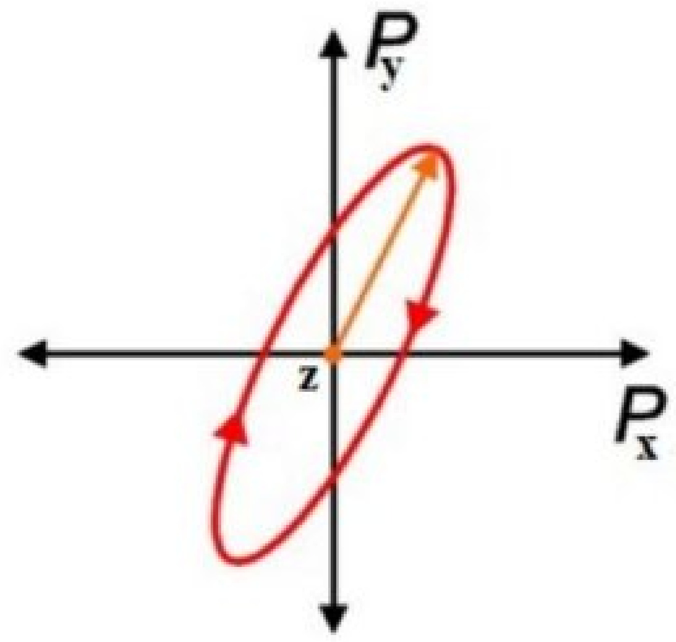
3. Elliptical Polarization



b

Elliptical polarization is a case where two waves of the same frequency, different phase, and different amplitude, combine into a single wave. The resulting vector P is the vector addition of the two field components P_x and P_y. The combined vector P appears to rotate in the pattern of an ellipse about the vector forming the direction of motion.
The condition for elliptical polarization for a wave propagating in z-direction is that, the electric field components must have a constant non-zero phase difference and their amplitudes unequal.

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